Definition and Description of a Global Spatial Data Model (GSDM)

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Introduction

Spatial data representing real world locations are three-dimensional (3-D). Modern measurement systems collect data in a three-dimensional environment. This paper defines and describes a global spatial data model (GSDM) which is a collection of mathematical concepts and procedures that can be used to collect, organize, store, process, manipulate, and evaluate 3-D spatial data more efficiently than can be done using a 2-dimensional conformal mapping model combined with 1-dimensional elevations. Measurements of quantities such as angles, length, time, current, mass, and temperature are used to determine spatial relationships which are stored for subsequent use and reuse. In the past, records of such measurements were written in field books, logs, or journals and the spatial information was compiled into an analog map which often served two purposes. The map could be both the primary storage medium for the spatial information while simultaneously being the end product of the data collection process. Spatial data are now collected, stored and manipulated digitally in an electronic environment and the primary storage medium is rarely the end product. Instead, the same digital data file can be duplicated repeatedly and used to generate and/or support many different spatial data products. In either case, whether developing an analog or digital spatial data product, algorithms are the mathematical rules used to manipulate measurements and spatial data to obtain meaningful spatial information. In addition, the quality of spatial information is dependent upon the quality of the original measurement, completeness of the required information and appropriateness of the algorithms used to manipulate the data. The GSDM includes both the algorithms for processing spatial data and mechanisms which can be used to provide a defensible statistical description of spatial data quality.

Acknowledgements

Credit for development of the individual concepts brought together in this collection belongs to many people over a long period of time. In formulating the GSDM, the goal was to begin with fundamental, almost self-evident, concepts and arrange them is ways which accommodate use of new technology while remaining consistent with established practice. Although innumerable persons including historical figures, professional colleagues, and students, have contributed feedback and insight to 3-D concepts, the following persons deserve specific recognition for their personal inspiration and professional encouragement:

- Ralph M. Berry - former surveying professor at the University of Michigan.
- Lassi A. Kivioja - retired geodesy professor from Purdue University.
- Edward M. Mikhail - professor of photogrammetry at Purdue University.
- John D. Bossler - Director of The Ohio State University Center for Mapping.
- Alfred Leick - author of the text, GPS Satellite Surveying.
- Kurt W. Bauer - former Executive Director of the Southeastern Wisconsin Regional Planning Commission.

The intent is to cite published works as appropriate. Undoubtedly sources were missed which could have been cited and some sources are cited which may not be the best or the primary source. Corrections/revisions will be made periodically.

Most of the concepts described herein were developed by others. For example, Appendix C in Bomford (1971) is titled "Cartesian Coordinates in Three Dimensions." Leick (1990 & 1995) defines the 3-D Geodetic Model, Mikhail (1976) provides a comprehensive discussion of functional and stochastic models, and when discussing models, Moritz (1978) comments on the simplicity of using the basic global rectangular X/Y/Z system without an ellipsoid. When the aforementioned concepts are combined in a systematic way with particular attention to the manner in which spatial data are used, the synergistic whole--the GSDM--appears to be greater than the sum of the parts.

Neither is the concept of a GSDM new. Seeber (1993) states that the concept of a global three-dimensional polyhedron network was purposed by H. Burns as early as 1878. The difference now is that the GPS and other modern technology have made a global network practical and the polyhedron need not be limited to earth-based points. It is further suggested that the GSDM is an appropriate model for describing the true instantaneous positions of a global network of continuously operating reference stations (CORS) computed in real time. An adopted mean position for each CORS will serve the needs of most users, but corrections for short term variations caused by earth tides and long term continental drift differences could be available to those needing them. It is acknowledged however, that a space-fixed inertial reference system is more appropriate for describing the motion of earth satellites.

The Global Spatial Data Model (GSDM)

The GSDM is a collection of mathematical concepts and procedures which can be used to manage spatial data both locally and globally. It consists of a functional model which describes the geometrical relationships and a stochastic model which describes the probabilistic characteristics--statistical qualities--of spatial data. The functional part of the model includes equations of geometrical geodesy and rules of solid geometry as related to various coordinate systems and is intended to be consistent with the 3-D Geodetic Model described by Leick (1990 & 1995) with the following exception; the GSDM, being strictly spatial, does not accommodate gravity measurements but presumes gravity affects are appropriately accommodated before data are entered into the spatial model. The stochastic portion of the GSDM is an application of concepts described by Mikhail (1976).

Although the GSDM described herein makes no attempt to accommodate non-Euclidean space or concepts, it does provide a simple universal foundation for many disparate coordinate systems used in various parts of the world and offers advantages of standardization for spatial data users in disciplines such as those listed in Figure 1. Hawking (1988) describes the search for a Grand Unification Theory (GUT) for the field of physics which will accommodate and adequately explain observations of physical phenomena from the very small to the extremely large. The extent to which the GSDM becomes the GUT of spatial data will be determined by its adoption and use world-wide.

The Functional Model Component

The functional model component of the GSDM is based upon a three-dimensional right-handed rectangular cartesian coordinate system with the origin located at the earth's center of mass. The X/Y plane lies in the equatorial plane with the X-axis at the 0° (Greenwich) meridian. The Z-axis coincides with the mean spin axis of the earth as defined by the Conventional Terrestrial Pole (Leick, 1990). This geocentric coordinate system is called an earth-centered earth-fixed (ECEF) coordinate system by the United States Defense Mapping Agency (DMA, 1987) and is widely used by many who work with global positioning system (GPS) and related data. Rules of solid geometry and vector algebra are universally applicable when working with ECEF coordinates and coordinate differences.

As shown in Figure 2A, the unique 3-dimensional position of any point on earth or near space is equivalently defined by traditional latitude/longitude/ellipsoid height coordinates or by a triplet of X/Y/Z coordinates expressed in meters. Due to the large distances involved, the X/Y/Z coordinate values can be quite large but personal computers (PC's) operating in double precision routinely handle 15 significant digits and 12 significant digits will accommodate all ECEF coordinate values within the "birdcage" of GPS satellites down to 0.1 mm. Some users may object to working with such large coordinate values but, as shown in Figure 2B, such objections will likely become inconsequential to the extent end user applications are designed to utilize coordinate differences (much smaller numbers and fewer digits).

Figure 3 is a GSDM schematic which illustrates relationships between the ECEF coordinate system and various other coordinate systems commonly used in connection with spatial data. A key feature on the diagram is a rotation matrix (Leick 1990 & others) used to convert $\Delta X/\Delta Y/\Delta Z$ coordinate differences to local $\Delta e/\Delta n/\Delta u$ coordinate differences at any point (local origin) specified by the user. Since a vector in 3-dimensional space is not altered by moving the origin or by changing the orientation of the reference coordinate system, a vector defined by its geocentric $\Delta X/\Delta Y/\Delta Z$ components is equivalently defined by local components and the rotation matrix is the mechanism which efficiently transforms a global perspective into a local one. The transpose of the rotation matrix is used to transform local components of a space vector to corresponding geocentric components.

Global Spatial Data Model - GSDM

(A Universal 3-D Model for Spatial Data)

The Global Spatial Data Model provides a simple, universal 3-dimensional mathematical foundation for the National Spatial Data Infrastructure (NSDI) which supports Geographic Information System (GIS) database applications in disciplines such as:

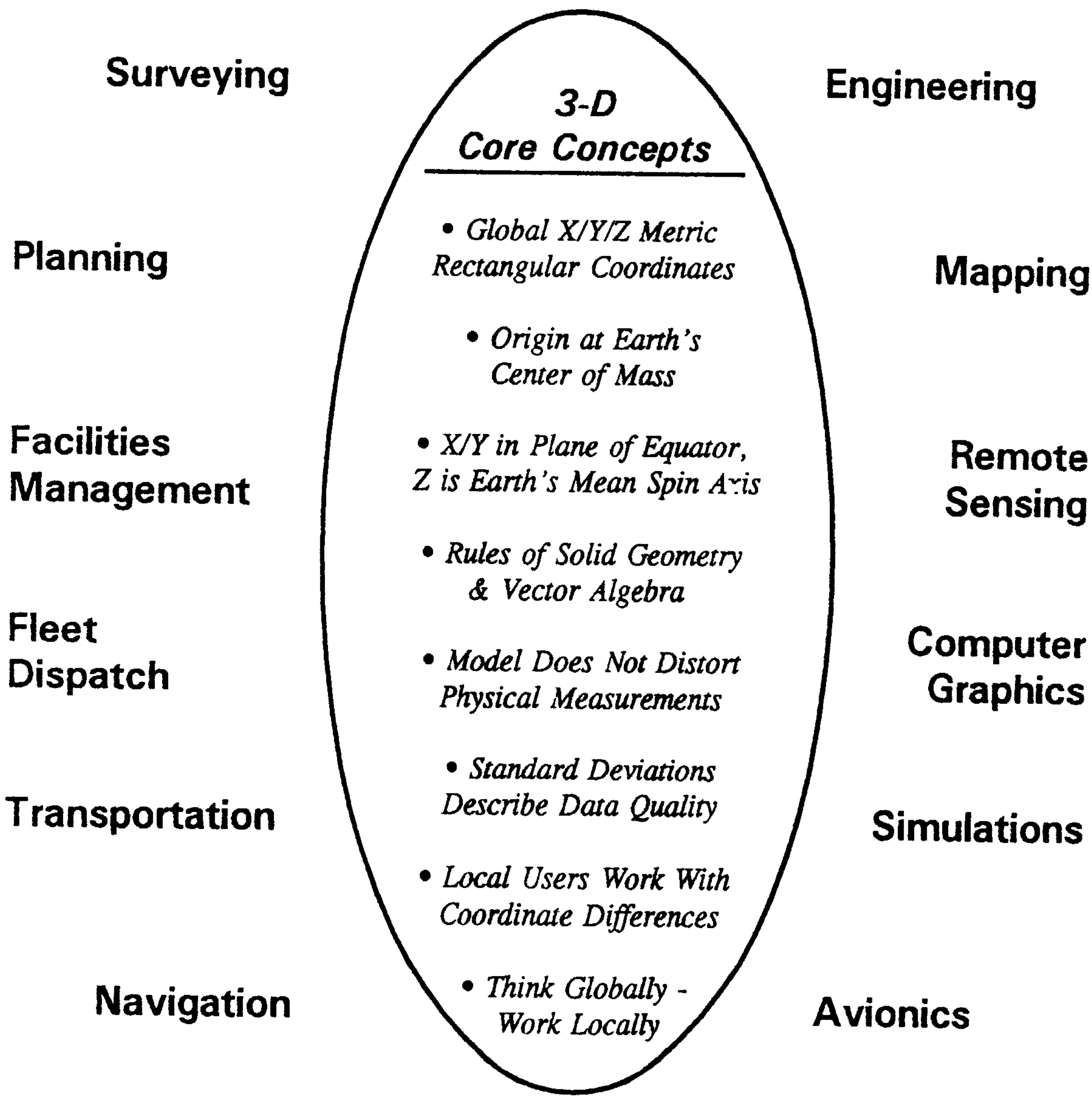


Figure 1, The Global Spatial Data Model (GSDM)

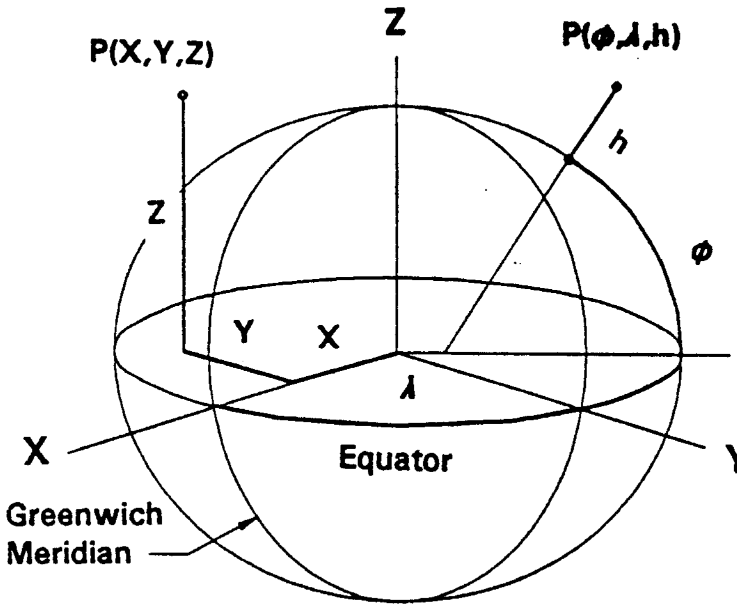


Figure 2A, Geocentric X/Y/Z and Geodetic φ/λ/h Coordinates

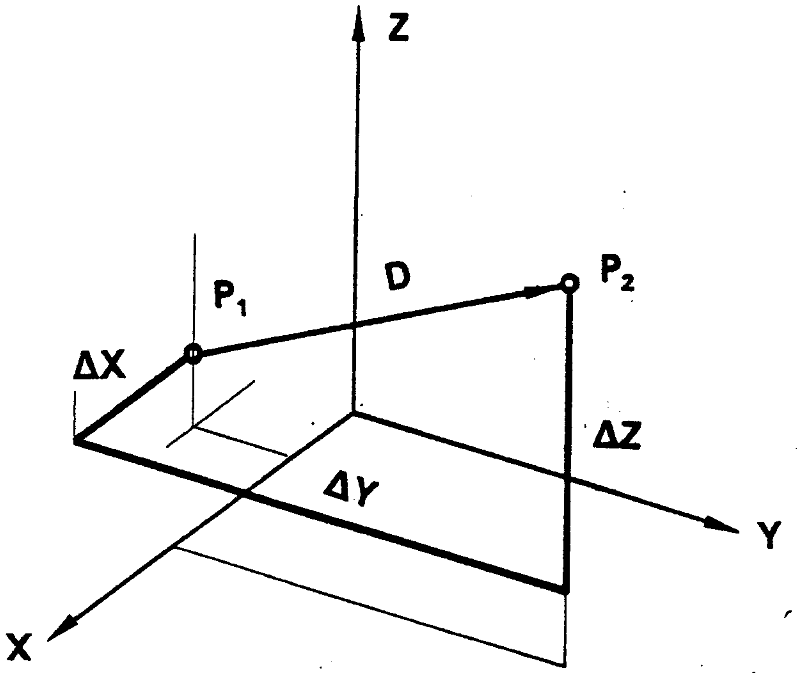


Figure 2B, GPS Technology Provides Precise ΔX/ΔY/ΔZ

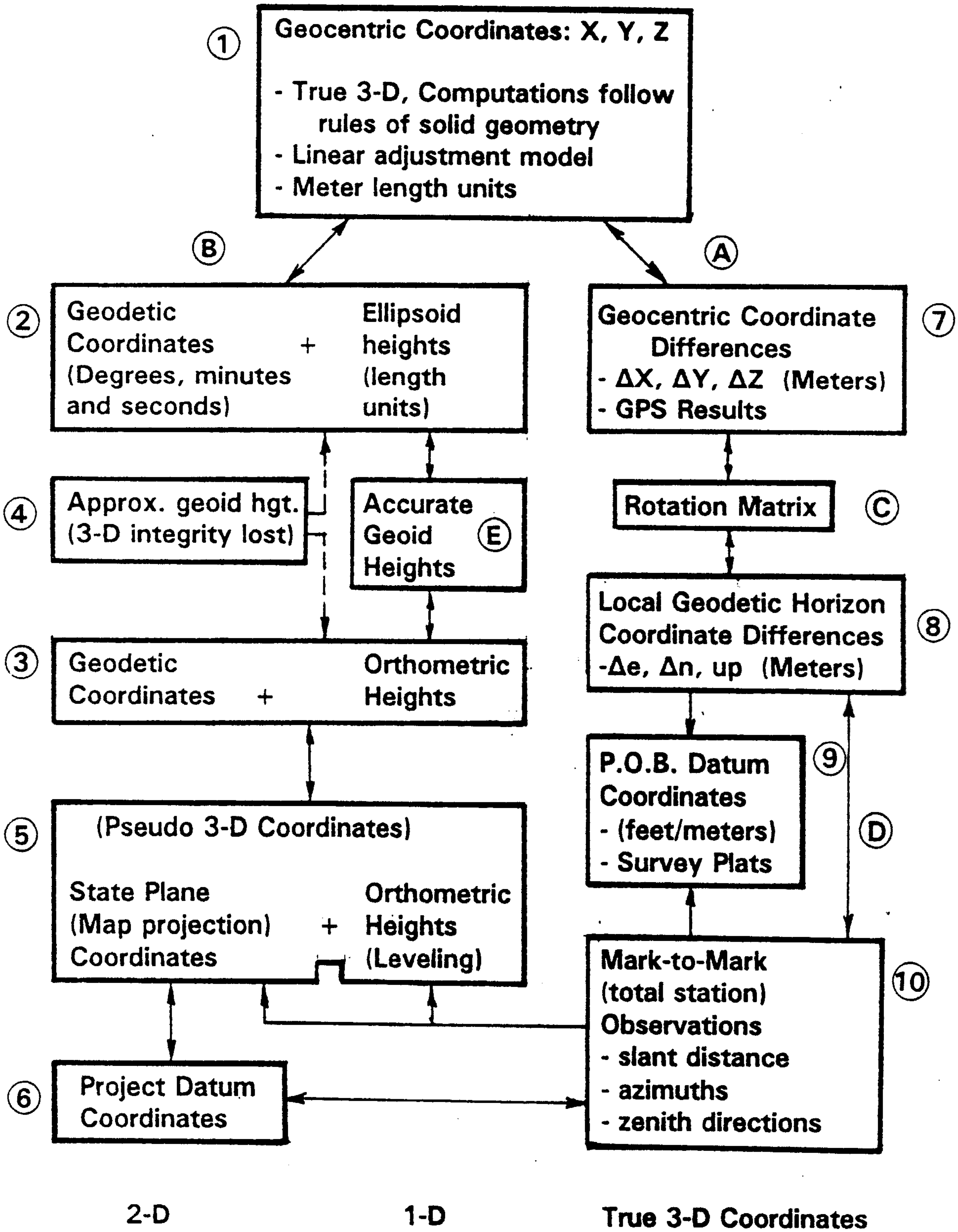


Figure 3, Schematic Showing Relationship of Coordinate Systems

With regard to Figure 3, the functional model includes equations for transforming spatial data described by coordinates in one numbered box to equivalent expression in a different coordinate system. A description of the numbered boxes is:

(1.) The geocentric X/Y/Z coordinates are the basis for all other coordinate values obtained from the GSDM. These are the defining values stored for each point in a digital spatial data file. Coordinate values in other coordinate systems are derived from the stored ECEF coordinates using algorithms which have been tested and proven accurate for a specified level of computational precision. Meter units, universal rules of solid geometry and vector algebra, and a linear adjustment model are features of working with this part of the GSDM.

(2.) Geodetic coordinates of latitude and longitude combined with ellipsoid height can define a three-dimensional position with the same precision and exactness as geocentric X/Y/Z coordinates. Equations are listed in a subsequent section by which coordinate values in one box can be converted to equivalent values in another. Use of angular sexagesimal units (degrees, minutes, and seconds) on the ellipsoid mixed with length units of meters for height makes 3-dimensional computation more complicated than when using ECEF rectangular coordinates.

(3.) Historically, horizontal coordinates of latitude and longitude have been combined with vertical elevations when mapping features on or near the earth's surface. The generic zero reference surface for elevation has been the geoid (or mean sea level) which admits to a physical definition but, as it turns out, is very difficult if not impossible to find. As a result (Zilkoski et al, 1992), an arbitrary reference surface which approximates, but does not define, mean sea level was selected for the North American Vertical Datum of 1988.

(4.) Geoid height is defined as the difference between ellipsoid height and elevation. With any two of the three elements known, the third can be found. If reliable ellipsoid height for a point (from GPS data) is combined with an appropriate geoid height (from geoid modeling), it is possible to obtain high quality orthometric height (elevations) from the GSDM. Appropriate use of standard deviations for the constituent components will provide a statistical assessment of the quality of such elevations.

(5.) Map projections were invented to address the challenge of representing a curved earth on a flat map. Conformal projections in particular have been used in surveying and mapping to precisely define a 2 dimensional relationship between latitude/longitude positions on the earth and equivalent plane coordinate positions on a flat map. The state plane coordinate systems implemented in the United States and the world-wide use of UTM coordinates represent systematic use of map projections.

It is important to note however, that elevations combined with map projection x/y (or north/east) plane coordinates is not an appropriate 3-dimensional rectangular model for two reasons:

a. Conformal projections are well defined in two dimensions only. There is no mathematical definition of elevation in conformal mapping.

b. The zero reference surface for elevation (approximated by sea level) is a non-regular curved surface. Three-dimensional rectangular integrity is preserved only so long as a flat earth can be safely assumed. These coordinates are therefore referred to a pseudo 3-D coordinates.

(6.) An important consideration when using state plane coordinates is the relationship of the grid inverse distance to actual ground-level horizontal distance. In applications such as highway centerline stationing the difference between grid and ground distance quickly becomes too great to ignore. Project datum coordinate systems were invented to accommodate the difference. Lack of standardization is an issue when considering project datum coordinates. For a summary of comments from 46 out of 50 state DOT's on the grid/ground distance difference, see Appendix III of Burkholder (1993).

(7.) Global positioning system (GPS) technology has been a driving force behind use of three-dimensional spatial data and helps create the demand for a GSDM. The primary output of a GPS survey is a 3-dimensional vector defined by its $\Delta X/\Delta Y/\Delta Z$ components. Because existing control stations were defined with geodetic coordinates of latitude and longitude (and other reasons), it was natural to continue building a 2-dimensional network using 3-dimensional measurements. And there certainly are cases where that practice can still be justified. But, the GSDM defines an environment in which the full value of three-dimensional data can be used to build high quality three-dimensional networks without being encumbered by many of the complex equations found in classical geodesy. Another benefit is that the stochastic model lends itself to implementation in the rectangular 3-dimensional environment more readily than in the latitude/longitude/height system.

(8.) The local geodetic horizon (Trimble 1990) is essentially the same as the local geodetic frame defined more precisely by Soler and Hothem (1988) and shares many similarities with local plane surveying practice. The primary difference is that "up" is defined by the ellipsoid normal instead of the plumb line. That difference is largely inconsequential except in cases where very high precision is required, the slope of the geoid is severe and many total station setups are required to make a survey tie between 3-D GPS (gravity-independent) control stations. Another difference with the GSDM is that the origin moves with the observer because one is working with local coordinate differences with respect to the user specified standpoint.

When working with the $\Delta e/\Delta n$ components, the horizontal distance is in the tangent plane through the standpoint and is the same horizontal distance plane surveyors have been using for generations. It is also the same as HD(1) as described in Burkholder (1991). Understandably, with a unique tangent plane at each standpoint, the tangent plane from Point A to Point B is slightly different that the tangent plane from Point B to Point A. But, geometrical integrity in 3 dimensions is preserved by the GSDM.

The azimuth from standpoint to forepoint obtained from arctan ($\Delta e/\Delta n$) very nearly equals the geodetic azimuth from standpoint to forepoint and is described more fully in Burkholder (1997). Most importantly, the GSDM gives the correct azimuth between each pair of points. The forward azimuth differs from the back azimuth due to convergence of the meridians. The GSDM competently provides the correct answer in each case.

(9) P.O.B. Datum Coordinates is a feature within the GSDM which accommodates long established local plane surveying practice without compromising geometrical integrity. P.O.B. Datum Coordinates permit the user to select any point in the data base as an origin. The 3-dimensional location of each additional point selected is listed with respect to the Point-of-Beginning (P.O.B.). Admittedly, this practice makes little sense for very large distances, but these local coordinate differences can be treated in the same manner as local plane coordinates and used on survey plats. Horizontal distances are in the tangent plane through the P.O.B. and azimuths are with respect to the meridian through the P.O.B. If surveys of adjacent tracts do not use the same P.O.B. there will be two azimuths for a common line (the difference is the amount of convergence between the two P.O.B.'s). However, if the P.O.B. is the same for both tracts, they will share a common basis of bearing - the meridian through the P.O.B.

Although suggested as a secondary means of obtaining elevation, the standard curvature and refraction correction, equation 5.7 of Davis, et al. (1981), can be combined with the "up" component to obtain elevation differences between standpoint and forepoint. The primary method for obtaining elevation relies on accurate geoid height and ellipsoid height.

(10) Spatial data measurements with conventional total station surveying instruments include slope distances, vertical (or zenith) angles, and determinations of bearings or azimuths. These measurements are used to compute local geodetic horizon coordinate differences of $\Delta e/\Delta n/\Delta u$. In reality the measurements are plumb line referenced while the GSDM stipulates the results be normal based. The difference is small, but important. Procedures for making Laplace corrections (current standard practice) can be implemented as required.

Equations for moving between various boxes are listed in the next section and keyed to the circled letters shown in Figure 3.

The Stochastic Model Component

The stochastic component of the GSDM is based upon storing the variance/covariance matrix associated with the geocentric X/Y/Z rectangular coordinates which define the location of each stored point. Standard variance/covariance propagation (Mikhail 1976) is used to determine the local east/north/up variance/covariance matrix of any point on an "as needed" basis by the user (this minimizes storage requirements). The same basic procedure is extended to other functional model computations and provides a statistically defensible method for tracking the influence of random errors to any derived quantity. In particular, the user can look at the standard deviation of a coordinate position (by individual component) in either the geocentric or local reference frame. The standard deviation of other derived quantities such as distance, azimuth, slope, area, or volume can be obtained using the same procedure for the functional model equations. An algorithm for 3-D coordinate computation and error propagation is given in Appendix A.

BURKORD™ - Software and Data Base

Prototype software for performing 3-dimensional coordinate geometry utilizing both the functional and stochastic components has been written and is called BURKORD™. An example of using BURKORD™ to perform standard 3-D computations is given in Appendix B. The term BURKORD™ applies specifically to the software which can be purchased from the author. BURKORD™ also applies as an adjective to describe a 3-dimensional data base built upon and utilizing the concepts and procedures described herein. A license to use the BURKORD™ name and the GSDM concepts as described herein to build a BURKORD™ Data Base can be purchased from Global COGO, Inc. of Circleville, Ohio.

Summary

The GSDM gives each user both control and responsibility. If bad information is used or if good information is used inappropriately, unreliable answers can be obtained. However, the opposite case is the important one. The GSDM defines a model and computational environment which can be used to manage spatial data efficiently. Each user has the option (control) of establishing criteria which must be met before spatial data can be used for a given purpose. The concept of Meta Data is important in establishing and preserving the credibility of spatial data (responsibility), but standard deviation (in any or all components) is a very efficient method for evaluating the quality of spatial data. Once the X/Y/Z position of a point is defined along with its variance/covariance matrix, the spatial data can be exchanged in a very compact format. The same solid geometry equations and error propagation are equally applicable world-wide and the mathematical procedures are already proven and accepted. The challenge is for the global community of spatial data users to discuss and reach consensus on details for implementation. However, the model is already defined and can be implemented immediately to build a BURKORD™ data base.

APPENDIX A

Algorithm for the Global Spatial Data Model

Functional Model:

The following symbols are defined and used as:

X/Y/Z	=	Geocentric right-handed rectangular coordinates
$\Delta X/\Delta Y/\Delta Z$	=	Geocentric coordinate differences
e/n/u	=	Local right-handed rectangular coordinates
$\Delta e/\Delta n/\Delta u$	=	Local coordinate differences
$\phi/\lambda/h$	=	Geodetic latitude/longitude (east) and ellipsoid height
a & b	=	Semi-major & semi-minor axes of reference ellipsoid
f	=	Flattening of reference ellipsoid
e^2	=	Eccentricity squared of reference ellipsoid; $e^2 = 2f - f^2$
N	=	Length of ellipsoid normal, also used for geoid height
r	=	Spatial distance from origin to point X/Y/Z
P	=	Projection of r to equatorial plane
a' b' h' ϕ'	=	Intermediate computational values used by Vincenty
T & U	=	Intermediate computational values used by Vincenty
S	=	Spatial slope distance between standpoint & forepoint
α	=	Geodetic azimuth at standpoint to forepoint
z or V	=	Zenith direction or vertical angle to forepoint
H	=	Orthometric height (elevation)
$\Delta N/\Delta h/\Delta H$	=	Changes in geoid, ellipsoid, and orthometric heights
c+r	=	Combined correction for curvature and refraction
HD(1)	=	Ground level horizontal distance, see Burkholder (1991)

Notes:

1. All distances are in units of meters.
2. Where two points are concerned, the standpoint is indicated by the subscript 1 while the forepoint is indicated by the subscript 2.

The following equations are keyed to the circled letters shown in Figure 3.

(A) Forward and Inverse Computations using geocentric coordinates:

Forward	Inverse	
$X_2 = X_1 + \Delta X$	$\Delta X = X_2 - X_1$	(1) & (2)
$Y_2 = Y_1 + \Delta Y$	$\Delta Y = Y_2 - Y_1$	(3) & (4)
$Z_2 = Z_1 + \Delta Z$	$\Delta Z = Z_2 - Z_1$	(5) & (6)

(B1) Convert geodetic latitude/longitude/ellipsoid height to geocentric X/Y/Z:

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{7}$$

$$X = (N + h) \cos\phi \cos\lambda \tag{8}$$

$$Y = (N + h) \cos\phi \sin\lambda \tag{9}$$

$$Z = (N [1 - e^2] + h) \sin\phi \tag{10}$$

(B2) Convert geocentric X/Y/Z to geodetic latitude/longitude/ellipsoid height:

It is difficult to invert the equations in B1 to obtain a closed form solution. A very good closed form approximation (which breaks down for very large values of ellipsoid height) is given on page 232 by Hofmann-Wellenhof et al (1992). Another option (page 225, Leick 1995) is to iterate equations (12) and (13) for an "exact" solution (assume N = 0 for first iteration and stop when h no longer changes by a significant amount).

$$\lambda = \tan^{-1}\left(\frac{Y}{X}\right) \tag{11}$$

$$\phi = \tan^{-1}\left[\frac{Z}{\sqrt{X^2 + Y^2}}\left(1 + \frac{e^2 N \sin\phi}{Z}\right)\right] \tag{12}$$

$$h = \frac{\sqrt{X^2 + Y^2}}{\cos\phi} - N \tag{13}$$

Another option is to use a precise "once through" approximation by Vincenty (1980) as:

$$b = a(1 - f) \tag{14}$$

$$P^2 = X^2 + Y^2, \qquad r^2 = P^2 + Z^2 \tag{15) \& (16}$$

$$h' = r - a + \frac{(a - b)Z^2}{r^2} \tag{17}$$

$$a' = a + h', \qquad b' = b + h' \tag{18) \& (19}$$

$$\tan\phi' = \left(\frac{a'}{b'}\right)^2 \left(\frac{Z}{P}\right) \left[1 + \frac{1}{4}\frac{e^4 h' a (Z^2 - P^2)}{a'^4}\right] \tag{20}$$

$$\cos^2\phi' = \frac{1}{1 + \tan^2\phi'}, \qquad \sin\phi' = \cos\phi' \tan\phi' \tag{21) \& (22}$$

$$T = \frac{(P - h'\cos\phi')^2}{a^2}, \qquad U = \frac{(Z - h'\sin\phi')^2}{b^2} \tag{23) \& (24}$$

$$h = h' + \frac{1}{2}\left[\frac{T + U - 1}{\frac{T}{a} + \frac{U}{b}}\right] \tag{25}$$

$$\phi = \tan^{-1}\left[\left(\frac{a}{b}\right)^2 \frac{(Z - e^2 h \sin\phi')}{P}\right] \tag{26}$$

$$\lambda = \tan^{-1}\left(\frac{Y}{X}\right) \tag{27}$$

Ⓒ The conversion between Geocentric Coordinate Differences and Local Geodetic Horizon Coordinate Differences can be accomplished very efficiently with a rotation matrix or the conversions can also be done using individual equations for each component. Both methods are presented.

C1. Geocentric Coordinate Differences can be converted to Local Coordinate Differences using the matrix form of equation (28) (Leick, 1995, Equations 7.9 and 7.10) or individually by component using equations (29), (30) and (31).

$$\begin{bmatrix}\Delta e\\ \Delta n\\ \Delta u\end{bmatrix} = \begin{bmatrix}-\sin\lambda & \cos\lambda & 0\\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi\\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi\end{bmatrix}\begin{bmatrix}\Delta X\\ \Delta Y\\ \Delta Z\end{bmatrix} \tag{28}$$

$$\Delta e = -\Delta X \sin\lambda + \Delta Y \cos\lambda \tag{29}$$

$$\Delta n = -\Delta X \sin\phi\cos\lambda - \Delta Y \sin\phi\sin\lambda + \Delta Z\cos\phi \tag{30}$$

$$\Delta u = \Delta X\cos\phi\cos\lambda + \Delta Y\cos\phi\sin\lambda + \Delta Z\sin\phi \tag{31}$$

C2. Local Geodetic Horizon Coordinate Differences can be converted to Geocentric Coordinate Differences in similar fashion using either the matrix form in equation (32) or individually by component using equations (33), (34), and (35). See Burkholder (1993).

$$\begin{bmatrix}\Delta X\\ \Delta Y\\ \Delta Z\end{bmatrix} = \begin{bmatrix}-\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda\\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda\\ 0 & \cos\phi & \sin\phi\end{bmatrix}\begin{bmatrix}\Delta e\\ \Delta n\\ \Delta u\end{bmatrix} \tag{32}$$

$$\Delta X = -\Delta e \sin\lambda - \Delta n \sin\phi\cos\lambda + \Delta u \cos\phi\cos\lambda \tag{33}$$

$$\Delta Y = \Delta e \cos\lambda - \Delta n \sin\phi\sin\lambda + \Delta u \cos\phi\sin\lambda \tag{34}$$

$$\Delta Z = \Delta n \cos\phi + \Delta u \sin\phi \tag{35}$$

(D) Local Geodetic Horizon Coordinate Differences are computed from terrestrial observations with equations (36), (37) and (38) (corrected as necessary for instrument calibration, atmospheric conditions, polar motion and local deflection-of-the-vertical. See Burkholder (1993)).

$$\Delta e = S \sin z \sin\alpha = HD(1) \sin\alpha \tag{36}$$

$$\Delta n = S \sin z \cos\alpha = HD(1)\cos\alpha \tag{37}$$

$$\Delta u = S \cos z \tag{38}$$

(E) Equations (1) through (38) follow rules of solid geometry and vector algebra and can be used to define and express spatial relationships either globally or locally without loss of geometrical integrity. However, (except for possible corrections due to deflection-of-the-vertical) gravity and level surfaces are not a part of the foregoing discussion. Given the importance of determining elevations (and the direction water will run), the orthometric height--elevation of a point--is obtained from geocentric coordinates by way of ellipsoid heights and geoid heights using equation (39).

$$H = h - N \tag{39}$$

At the risk of oversimplifying a complex issue, equation (39) is very useful for computing elevations, but it presumes accurate geoid heights are known. In reality, a better method is to use a known elevation at point 1 along with observed ellipsoid height difference from GPS measurements and modeled geoid height difference from a model such as GEOID96. In that case, the elevation of point 2 is:

$$H_2 = H_1 + \Delta H = H_1 + \Delta h - \Delta N \tag{40}$$

$$H_2 = H_1 + (h_2 - h_1) - (N_2 - N_1) \tag{41}$$

Equations (40) and (41) are equivalent and very useful, but still limited by the accuracy of available information. The prudent user understands that in all cases, the value of the least accurate of the 3 elements in equation (39) should be computed from the other two more reliable elements. The trend being driven by current technology and ongoing research is to compute orthometric heights from ellipsoid heights and geoid heights.

In cases where the forepoint elevation is to be computed from the "up" component of the local geodetic horizon coordinates, the curvature and refraction correction can be used locally to approximate H_2 as:

$$H_2 = H_1 + \Delta H = H_1 + \Delta u + (c + r) \tag{42}$$

$$H_2 = H_1 + \Delta u + 0.0675 \frac{(\Delta e^2 + \Delta n^2)}{1{,}000{,}000}$$ Equation 5.7, Davis, et al. (1981) (43)

Stochastic Model:

The equations listed in this section represent an application of the laws of variance/covariance propagation as described in Chapter 4 of Mikhail (1976) and make extensive use of the following matrix formulation applied to equations of the functional model:

$$\Sigma_{YY} = J_{YX}\, \Sigma_{XX}\, J^t_{YX} \tag{44}$$

where:

Σ_{YY} = Covariance matrix of computed result.
Σ_{XX} = Covariance matrix of variables used in computation.
J_{YX} = Jacobian matrix of partial derivatives of the result with respect to the variables.

In particular, the following symbols are in addition to those used for the functional model listed previously:

$\sigma_X^2\ \sigma_Y^2\ \sigma_Z^2$ = Variances of geocentric coordinates for a point.

$\sigma_{XY}\ \sigma_{XZ}\ \sigma_{YZ}$ = Covariances of geocentric coordinates for a point.

$\sigma_e^2\ \sigma_n^2\ \sigma_u^2$ = Variances of a point in the local reference frame.

$\sigma_{en}\ \sigma_{eu}\ \sigma_{nu}$ = Covariances of a point in the local reference frame.

$\sigma_{\Delta X}^2\ \sigma_{\Delta Y}^2\ \sigma_{\Delta Z}^2$ = Variances of geocentric coordinate differences.

$\sigma_{\Delta X \Delta Y}\ \sigma_{\Delta X \Delta Z}\ \sigma_{\Delta Y \Delta Z}$ = Covariances of geocentric coordinate differences.

$\sigma_{\Delta e}^2\ \sigma_{\Delta n}^2\ \sigma_{\Delta u}^2$ = Variances of coordinate differences in local frame.

$\sigma_{\Delta e \Delta n}\ \sigma_{\Delta e \Delta u}\ \sigma_{\Delta n \Delta u}$ = Covariances of coordinate differences in local frame.

$\sigma_S^2\ \sigma_\alpha^2$ = Variances of local horizontal distance and azimuth.

$\sigma_{S\alpha}$ = Covariance of local horizontal distance with azimuth.

σ_z^2 = Variance of zenith direction.

The stochastic information for each point is stored as its geocentric covariance matrix.

A. The covariance matrix is symmetric 3 X 3. Six numbers are required to store upper (or lower) triangular values.

B. Units in the covariance matrix is meters squared.

C. Standard deviation is square root of diagonal elements.

$$\begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z^2 \end{bmatrix}$$

Functional model computations supported by the stochastic model include:

A. Geocentric coordinate differences from geocentric coordinates:

Matrix formulation of the functional model equations is:

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \tag{45}$$

The Jacobian matrix noted above is used with the general matrix variance/covariance propagation formulation, equation (44) as:

$$\Sigma_\Delta = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1Y_1} & \sigma_{X_1Z_1} \\ \sigma_{X_1Y_1} & \sigma_{Y_1}^2 & \sigma_{Y_1Z_1} \\ \sigma_{X_1Z_1} & \sigma_{Y_1Z_1} & \sigma_{Z_1}^2 \end{bmatrix} & \begin{bmatrix} \sigma_{X_1X_2} & \sigma_{X_1Y_2} & \sigma_{X_1Z_2} \\ \sigma_{Y_1X_2} & \sigma_{Y_1Y_2} & \sigma_{Y_1Z_2} \\ \sigma_{Z_1X_2} & \sigma_{Z_1Y_2} & \sigma_{Z_1Z_2} \end{bmatrix} \\ \begin{bmatrix} \sigma_{X_1X_2} & \sigma_{Y_1X_2} & \sigma_{Z_1X_2} \\ \sigma_{X_1Y_2} & \sigma_{Y_1Y_2} & \sigma_{Z_1Y_2} \\ \sigma_{X_1Z_2} & \sigma_{Y_1Z_2} & \sigma_{Z_1Z_2} \end{bmatrix} & \begin{bmatrix} \sigma_{X_2}^2 & \sigma_{X_2Y_2} & \sigma_{X_2Z_2} \\ \sigma_{Y_2X_2} & \sigma_{Y_2}^2 & \sigma_{Y_2Z_2} \\ \sigma_{Z_2X_2} & \sigma_{Z_2Y_2} & \sigma_{Z_2}^2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

Assuming no correlation between points and omitting many computational details, the result is:

$$\Sigma_\Delta = \begin{bmatrix} \sigma^2_{\Delta X} & \sigma_{\Delta X \Delta Y} & \sigma_{\Delta Y \Delta Z} \\ \sigma_{\Delta X \Delta Y} & \sigma^2_{\Delta Y} & \sigma_{\Delta Y \Delta Z} \\ \sigma_{\Delta X \Delta Z} & \sigma_{\Delta Y \Delta Z} & \sigma^2_{\Delta Z} \end{bmatrix} = \begin{bmatrix} \left(\sigma^2_{X_1} + \sigma^2_{X_2}\right) & \left(\sigma_{X_1Y_1} + \sigma_{X_2Y_2}\right) & \left(\sigma_{X_1Z_1} + \sigma_{X_2Z_2}\right) \\ \left(\sigma_{X_1Y_1} + \sigma_{X_2Y_2}\right) & \left(\sigma^2_{Y_1} + \sigma^2_{Y_2}\right) & \left(\sigma_{Y_1Z_1} + \sigma_{Y_2Z_2}\right) \\ \left(\sigma_{X_1Z_1} + \sigma_{X_2Z_2}\right) & \left(\sigma_{Y_1Z_1} + \sigma_{Y_2Z_2}\right) & \left(\sigma^2_{Z_1} + \sigma^2_{Z_2}\right) \end{bmatrix} \quad (47)$$

B. Local coordinate differences from geocentric coordinate differences:

The matrix formulation of the functional model equations is:

$$\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \quad (48)$$

The Jacobian matrix noted above is used with the general error propagation formulation to get the covariance matrix of local coordinate differences as:

$$\Sigma_\Delta = \begin{bmatrix} \sigma^2_{\Delta e} & \sigma_{\Delta e \Delta n} & \sigma_{\Delta e \Delta u} \\ \sigma_{\Delta e \Delta n} & \sigma^2_{\Delta n} & \sigma_{\Delta n \Delta u} \\ \sigma_{\Delta e \Delta u} & \sigma_{\Delta n \Delta u} & \sigma^2_{\Delta u} \end{bmatrix} = J \begin{bmatrix} \sigma^2_{\Delta X} & \sigma_{\Delta X \Delta Y} & \sigma_{\Delta X \Delta Z} \\ \sigma_{\Delta X \Delta Y} & \sigma^2_{\Delta Y} & \sigma_{\Delta Y \Delta Z} \\ \sigma_{\Delta X \Delta Z} & \sigma_{\Delta Y \Delta Z} & \sigma^2_{\Delta Z} \end{bmatrix} J^t \quad (49)$$

C. Geocentric coordinate differences from local coordinate differences:

The matrix formulation of the functional model equations is:

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} \quad (50)$$

The Jacobian matrix noted above is used with the general error propagation formulation to get the covariance matrix of geocentric coordinate differences as:

$$\Sigma_{\Delta} = \begin{bmatrix} \sigma^2_{\Delta X} & \sigma_{\Delta X \Delta Y} & \sigma_{\Delta X \Delta Z} \\ \sigma_{\Delta X \Delta Y} & \sigma^2_{\Delta Y} & \sigma_{\Delta Y \Delta Z} \\ \sigma_{\Delta X \Delta Z} & \sigma_{\Delta Y \Delta Z} & \sigma^2_{\Delta Z} \end{bmatrix} = J \begin{bmatrix} \sigma^2_{\Delta e} & \sigma_{\Delta e \Delta n} & \sigma_{\Delta e \Delta u} \\ \sigma_{\Delta e \Delta n} & \sigma^2_{\Delta n} & \sigma_{\Delta n \Delta u} \\ \sigma_{\Delta e \Delta u} & \sigma_{\Delta n \Delta u} & \sigma^2_{\Delta u} \end{bmatrix} J^t \qquad (51)$$

D. A rigorous transformation from one 3-dimensional rectangular coordinate system to another is given by a seven-parameter transformation. In matrix form, the functional model equation is:

$$\mathbf{X}_2 = s\,\mathbf{R}\,\mathbf{X}_1 + \mathbf{K} \quad \text{where} \qquad (52)$$

$\mathbf{X}_2$ = Vector of frame 2 coordinates
s = Scaler = 1.0
$\mathbf{R}$ = Rotation matrix, frame 1 to frame 2
$\mathbf{X}_1$ = Vector of frame 1 coordinates
$\mathbf{K}$ = Translation vector

Applying covariance propagation to that system of equations gives:

$$\Sigma_{YY} = \mathbf{J}\,\Sigma_{XX}\,\mathbf{J}^t \quad \text{where} \qquad (53)$$

Σ_{YY} = Covariance matrix of frame 2 coordinates
Σ_{XX} = Covariance matrix of frame 1 coordinates
$\mathbf{J}$ = Partial derivative matrix of frame 2 coordinates with respect to frame 1. (Rotation matrix, $\mathbf{R}_1$ or $\mathbf{R}_2$, see below)

Therefore, the covariance matrix of a point position in the local reference frame is obtained from the covariance matrix of the same point in the geocentric reference frame as:

$$\Sigma_{enu} = \begin{bmatrix} \sigma^2_e & \sigma_{en} & \sigma_{eu} \\ \sigma_{en} & \sigma^2_n & \sigma_{nu} \\ \sigma_{eu} & \sigma_{nu} & \sigma^2_u \end{bmatrix} = R_1 \begin{bmatrix} \sigma^2_X & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & \sigma^2_Y & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & \sigma^2_Z \end{bmatrix} R_1^t \quad \text{where} \qquad (54)$$

$$R_1 = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix}$$

And, the covariance matrix of a point position in the geocentric reference frame is obtained from the covariance matrix of the same point in the local reference frame as:

$$\Sigma_{XYZ} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z^2 \end{bmatrix} = R_2 \begin{bmatrix} \sigma_e^2 & \sigma_{en} & \sigma_{eu} \\ \sigma_{en} & \sigma_n^2 & \sigma_v \\ \sigma_{eu} & \sigma_v & \sigma_u^2 \end{bmatrix} R_2^t \quad \text{where} \tag{55}$$

$$R_2 = \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix}$$

E. Inverse distance and azimuth from local coordinate differences:

The functional model equations for distance and azimuth are:

$$S = \sqrt{\Delta e^2 + \Delta n^2} \tag{56}$$

$$\alpha = \tan^{-1}\left(\frac{\Delta e}{\Delta n}\right) \tag{57}$$

The Jacobian matrix of partial derivatives is:

$$J = \begin{bmatrix} \frac{\partial S}{\partial \Delta e} & \frac{\partial S}{\partial \Delta n} & \frac{\partial S}{\partial \Delta u} \\ \frac{\partial \alpha}{\partial \Delta e} & \frac{\partial \alpha}{\partial \Delta n} & \frac{\partial \alpha}{\partial \Delta u} \end{bmatrix} = \begin{bmatrix} \frac{\Delta e}{S} & \frac{\Delta n}{S} & 0 \\ \frac{\Delta n}{S^2} & \frac{\Delta e}{S^2} & 0 \end{bmatrix} \tag{58}$$

Using the covariance propagation formulation, the results are:

$$\Sigma_{INV} = \begin{bmatrix} \sigma_S^2 & \sigma_{S\alpha} \\ \sigma_{S\alpha} & \sigma_\alpha^2 \end{bmatrix} = J \begin{bmatrix} \sigma_{\Delta e}^2 & \sigma_{\Delta e \Delta n} & \sigma_{\Delta e \Delta u} \\ \sigma_{\Delta e \Delta n} & \sigma_{\Delta n}^2 & \sigma_{\Delta n \Delta u} \\ \sigma_{\Delta e \Delta u} & \sigma_{\Delta n \Delta u} & \sigma_{\Delta u}^2 \end{bmatrix} J^t \tag{59}$$

F. For a new point based upon a traverse computation during which the geocentric coordinates of the point are determined, the functional model in matrix form is:

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \qquad (60)$$

The covariance matrix of the variables (which assumes no correlation between the coordinates of Point 1 and the geocentric coordinate differences) is:

$$\Sigma_{variables} = \begin{bmatrix} \begin{bmatrix} \sigma^2_{X_1} & \sigma_{X_1Y_1} & \sigma_{X_1Z_1} \\ \sigma_{X_1Y_1} & \sigma^2_{Y_1} & \sigma_{Y_1Z_1} \\ \sigma_{X_1Z_1} & \sigma_{Y_1Z_1} & \sigma^2_{Z_1} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \sigma^2_{\Delta X} & \sigma_{\Delta X\Delta Y} & \sigma_{\Delta X\Delta Z} \\ \sigma_{\Delta X\Delta Y} & \sigma^2_{\Delta Y} & \sigma_{\Delta Y\Delta Z} \\ \sigma_{\Delta X\Delta Z} & \sigma_{\Delta Y\Delta Z} & \sigma^2_{\Delta Z} \end{bmatrix} \end{bmatrix} \qquad (61)$$

Applying covariance propagation, equation (44), the covariance matrix of a newly established point is:

$$\Sigma_{XYZ_2} = J\, \Sigma_{variables}\, J^t = \Sigma_{XYZ_1} + \Sigma_{\Delta X\Delta Y\Delta Z} \qquad (62)$$

Note that the covariance matrix for Point 1 is either presumed known or zero.

The remaining portion of this section is addressed to obtaining the covariance matrix of the geocentric coordinate differences. Two identifiable options are:

1. Using GPS processing results, the variance/covariance values of the geocentric coordinate differences for a base line are available and used.

2. The geocentric coordinate differences of the vector from Point 1 to Point 2 is obtained by rotating local coordinate differences to geocentric coordinate differences using equation (32). The covariance matrix of the geocentric coordinate differences is obtained from the covariance matrix of the local coordinate differences using equation (51).

The next question addressed is that of obtaining the covariance matrix of local coordinate differences from conventional "total station" surveying measurements. An underlying assumption[1] here (which is nearly true, but not quite) is that the azimuth of each line is an independent quantity. The functional model for local (mark to mark) coordinate differences is:

$$\begin{bmatrix} \Delta e \\ \Delta n \\ \Delta u \end{bmatrix} = \begin{bmatrix} S \sin z \sin\alpha \\ S \sin z \cos\alpha \\ S \cos z \end{bmatrix} \quad \text{where} \qquad (63)$$

S = Slope distance, standpoint to forepoint.
α = Azimuth, standpoint to forepoint.
z = Zenith direction, standpoint to forepoint.
(If reciprocal zenith directions are used, the "curvature" portion of the correction should be removed.)

The Jacobian matrix is obtained as the matrix of partial derivatives with respect to the observed quantities as:

$$J = \begin{bmatrix} \frac{\partial \Delta e}{\partial S} & \frac{\partial \Delta e}{\partial z} & \frac{\partial \Delta e}{\partial \alpha} \\ \frac{\partial \Delta n}{\partial S} & \frac{\partial \Delta n}{\partial z} & \frac{\partial \Delta n}{\partial \alpha} \\ \frac{\partial \Delta u}{\partial S} & \frac{\partial \Delta u}{\partial z} & \frac{\partial \Delta u}{\partial \alpha} \end{bmatrix} = \begin{bmatrix} \sin z \sin\alpha & S\cos z \sin\alpha & S\sin z \cos\alpha \\ \sin z \cos\alpha & S\cos z \cos\alpha & -S\sin z \sin\alpha \\ \cos z & -S\sin z & 0 \end{bmatrix} \qquad (64)$$

---

[1] The same assumption has been widely adopted whenever the Compass Rule is used to adjust a traverse whose azimuths were determined with a transit or theodolite instead of a compass or gyroscope. Although formulation of the equations gets tedious and more storage is required for larger matrices, the stochastic model will competently handle correlation between points and correlation from one course to another.

The variance/covariance matrix of the observed quantities (variables) is a diagonal matrix of variances due to independence of the measurements. This assumption is related to, but not the same as the earlier assumption of independence of azimuth from course to course. Using the Jacobian matrix and the variance matrix of observations in equation (44), the covariance matrix of local coordinate differences is:

$$\Sigma_{\Delta e \Delta n \Delta u} = \begin{bmatrix} \sigma^2_{\Delta e} & \sigma_{\Delta e \Delta n} & \sigma_{\Delta e \Delta u} \\ \sigma_{\Delta e \Delta n} & \sigma^2_{\Delta n} & \sigma_{\Delta n \Delta u} \\ \sigma_{\Delta e \Delta u} & \sigma_{\Delta n \Delta u} & \sigma^2_{\Delta u} \end{bmatrix} = J \begin{bmatrix} \sigma^2_S & 0 & 0 \\ 0 & \sigma^2_z & 0 \\ 0 & 0 & \sigma^2_\alpha \end{bmatrix} J^t \qquad (65)$$

Options for using the stochastic model are:

A. The stochastic model is not used for a point:

1. No standard deviation or covariance values are input.
2. The covariance matrix is set to zero.
3. The geocentric X/Y/Z position is used as being errorless.

B. The geocentric covariance matrix for a point is defined at the same time as its X/Y/Z coordinates:

1. Input as standard deviations of X/Y/Z:

   a. Units of meters input (stored as variance - meters squared)
   b. No correlation data input. Off diagonal elements are zero.
   c. Different components may have different values.

2. Full covariance matrix is input for each point:

   a. Units of meters squared
   b. Six elements input and stored
   c. Caution - care is required to assure input of covariance values which are mathematically consistent. If inappropriate values of covariance are input, negative variances (incorrect) in another reference frame may be the result.

C. The geocentric covariance matrix for a point is computed from the local covariance matrix which is determined at the same time the point is defined. Input options for local covariance matrix are:

1. Input as standard deviations of e/n/u:

    a. Units of meters input (stored as variance - meters squared)
    b. No correlation data input. Off diagonal elements are zero.
    c. Different components may have different values.

2. Full local covariance matrix is input for each point:

    a. Units of meters squared
    b. Six elements input and stored
    c. Caution - care is required to assure input of covariance values which are mathematically consistent. If inappropriate values of covariance are input, negative variances (incorrect) in another reference frame may be the result.

3. The geocentric covariance matrix for the point defined is computed using equation (55).

D. Point 2 is established by adding user supplied geocentric coordinate differences to the geocentric coordinates of Point 1. The covariance matrix of Point 2 is found using equation (62). Options for obtaining the covariance matrix of the geocentric coordinate differences include:

1. No covariance data are available.
    a. No covariance values are input.
    b. Covariance matrix of geocentric coordinate differences is set to zero.
    c. The uncertainty of Point 2 is the same as at Point 1.

2. Input standard deviations of geocentric coordinate differences.
    a. Units of meters input (stored as variance - meters squared)
    b. No correlation data input. Off diagonal elements are zero.
    c. Different components may have different values.

3. Full covariance matrix is input for geocentric coordinate differences of the vector:
    a. Units of meters squared
    b. Six elements input and stored
    c. Caution - care is required to assure input of covariance values which are mathematically consistent. If inappropriate values of covariance are input, negative variances (incorrect) in another reference frame may be the result.

E. Point 2 is established by adding geocentric coordinate differences, obtained by rotating local coordinate differences to the geocentric reference frame, to the geocentric coordinates of Point 1. In this case, equation (51) is used before the covariance matrix of Point 2 can be found using equation (62). Options for obtaining the covariance matrix of the local coordinate differences include:

1. No covariance data are available.
   a. No covariance values are input.
   b. Covariance matrix of local coordinate differences is set to zero.
   c. The uncertainty of Point 2 is the same as at Point 1.

2. Input standard deviations of local coordinate differences.
   a. Units of meters input (stored as variance - meters squared)
   b. No correlation data input. Off diagonal elements are zero.
   c. Different components may have different values.

3. Full covariance matrix is input for geocentric coordinate differences of the vector:
   a. Units of meters squared
   b. Six elements input and stored
   c. Caution - care is required to assure input of covariance values which are mathematically consistent. If inappropriate values of covariance are input, negative variances (incorrect) in another reference frame may be the result.

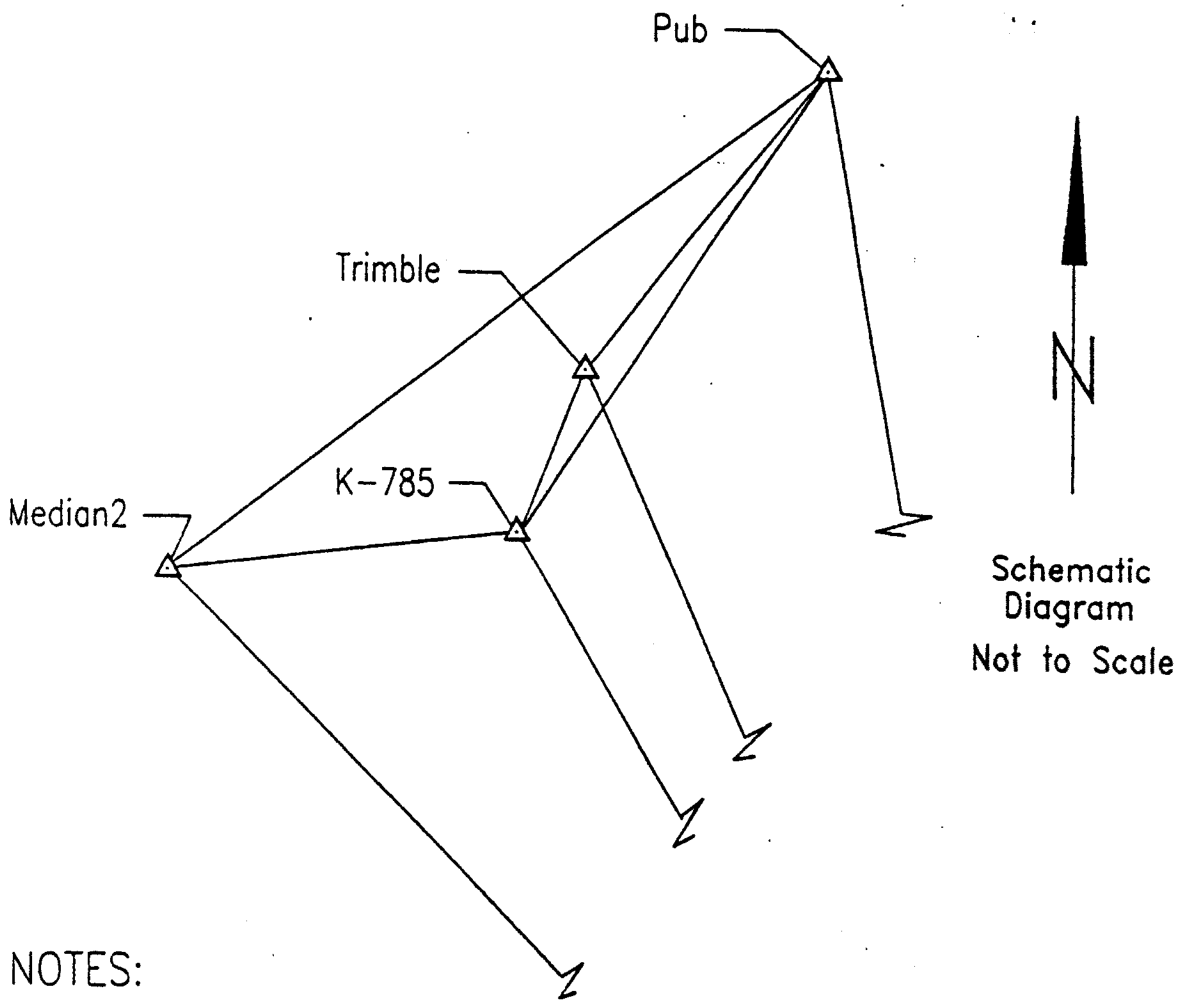
4. Use equation (65) to obtain the covariance matrix of the local coordinate differences based upon independent measurements of slope distance, zenith directions, and azimuth.
   a. Only standard deviations are input. Units are:
      1. Meters for slope distance.
      2. Radians for angular values. Programs can be written to accept other units of input.
   b. No correlation data input. Off diagonal elements are zero.
   c. Each independent observation has its own standard deviation.
   d. If vertical angles are used, two options are:
      1. Change equations (63) and (64).
      2. Compute zenith direction from vertical angle. (Standard deviation is same for vertical or zenith.)

APPENDIX B

3-D Example Using BURKORD™ at Oregon's Institute of Technology

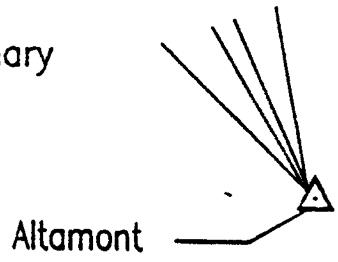
Klamath Falls, Oregon

COMPARISON OF COORDINATE SYSTEMS
GPS Points at Oregon Tech



NOTES:

1. Station "Altamont" located off campus and part of Oregon High Precision Geodetic Network (HPGN).
2. Stations "Pub", "Trimble", "K-785" and "Median2" are located on campus of Oregon Tech.
3. Campus network is also tied to California HPGN. Connection is not shown.
4. Geocentric (3-D) coordinates are primary data for all campus points.



Title: Control Points for Comparison of Local & 3-D Coordinate Systems		
OREGON INSTITUTE OF TECHNOLOGY	Date: Nov. 1992	File: OIT-GPS.DWG
Klamath Falls, Oregon 97601-8801	Drawn by: Earl F. Burkholder	

3-D COGO Example Data - GPS Based Control Points (HPGN) at Oregon Institute of Technology

Data Collected by: NGS and various vendors during on campus demonstrations

Computed by: Earl F. Burkholder & OIT Geodesy Students - 1992 and 1993

Tabulated and Printed by: Earl F. Burkholder - October, 1996
Global COGO Inc.
Circleville, Ohio

Station	Geocentric Coordinates	Geodetic Coordinates	Assumed Standard Deviations Example 1	Example 2
K-785	X= -2,490,977.048 Y= -4,019,738.188 Z= 4,267,460.384	ϕ= 42 15 16.99294N λ=121 47 09.35422W h= 1,297.866	0.005 N/S 0.005 E/W 0.005 UP	0.10 N/S 0.10 E/W 0.005 UP
Trimble	X= -2,490,854.501 Y= -4,019,681.242 Z= 4,267,591.406	ϕ= 42 15 22.59644N λ=121 47 06.11898W h= 1,302.365	0.005 N/S 0.005 E/W 0.10 UP	0.10 N/S 0.10 E/W 0.005 UP
Median-2	X= -2,491,313.163 Y= -4,019,556.682 Z= 4,267,423.420	ϕ= 42 15 15.61009N λ=121 47 25.98592W h= 1,289.871	0.100 N/S 0.100 E/W 0.100 UP	1.00 N/S 1.00 E/W 0.005 UP
Pub	X= -2,490,534.863 Y= -4,019,658.196 Z= 4,267,850.838	ϕ= 42 15 32.91354N λ=121 46 54.79688W h = 1,337.720	0.005 N/S 0.005 E/W 0.100 UP	1.00 N/S 1.00 E/W 1.00 UP

Notes:

1. All linear units are meters.
2. Standard deviations have been assumed for all examples. Other standard deviation combinations (or covariance information) can be chosen by the user and used in a demonstration version of BURKORD, a 3-D coordinate geometry software package.
3. Three separate printouts shows examples based on these data.
4. Example 1 shows precise vertical on 1 point, precise horizontal on 3 of 4 points, one "weak" point, and one "strong" point.
5. Example 2 shows precise vertical on 3 of 4 points and approximate horizontal on all points - one point very approximate, 1 meter.
6. Example 3 starts on K-785 as the "known" point and uses the values shown below for 4 modes of traversing; 1) geocentric coordinate differences, 2) local coordinate differences, 3) slope distance, vertical angle, azimuth, and 4) slope distance, zenith direction, azimuth.

K-785 to Trimble:

ΔX = 122.5471 +/- 0.006
ΔY = 56.9460 +/- 0.004
ΔZ = 131.0224 +/- 0.005

K-785 to Median-2:

SD = 383.776 +/- 0.005
VA = -1° 11' 44" +/- 10"
Azi = 263° 36' 56" +/- 5"

Trimble to Pub:

ΔE = 259.5629 +/- 0.004
ΔN = 318.4064 +/- 0.006
ΔU = 35.3414 +/- 0.008

Median-2 to Pub-2:

SD = 893.7231 +/- 0.005
Zen = 86° 56' 06" +/- 10"
Azi = 53° 14' 38" +/- 5"

'BURKORD(TM)' COMPUTES 3-D COORDINATE GEOMETRY POSITIONS FOR SPATIAL DATA UTILIZING GPS VECTORS, LOCAL COORDINATE DIFFERENCES AND 3-D SURVEYING MEASUREMENTS.


```
USER:   EARL F. BURKHOLDER
DATE:   JANUARY 12, 1997

PROGRAM:         BURKORD(TM) - VERSION 8A, DECEMBER 1996  S/N 8AB96000
DATA FILE:       EXMPL-8A.DAT
OUTPUT FILE:     TAPS-TST.OUT

CLIENT/AGENCY:   TEST OF BURKORD(TM) VERSION 8A BY GLOBAL COGO, INC.
JOB/PROJECT:     HARN CONNECTED GPS POINTS AT OREGON INSTITUTE OF TECHNOLOGY
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* *

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A LISTING OF POINTS IN ACTIVE PROJECT IS:
  101 -2490977.0480 -4019738.1880  4267460.3840 .000025 .000025 .000025 .000000 .000000 .000000 K-785
  102 -2490854.5010 -4019681.2420  4267591.4060 .001541 .003973 .004536 .002447-.002615-.004220 TRIMBLE
  103 -2491313.1630 -4019556.6820  4267423.4200 .010000 .010000 .010000 .000000 .000000 .000000 MEDIAN-2
  104 -2490534.8630 -4019658.1960  4267850.8380 .001541 .003973 .004536 .002446-.002615-.004220 PUB

AN EXPANDED LISTING OF POINTS   101 TO   104

  101     K-785                                  X         Y         Z               E         N         U
LAT (N+S-)   42 15 16.992928 X: -2490977.0480  X  .25E-04                       E  .25E-04
LON (E+W-) -121 47  9.354217 Y: -4019738.1880  Y  .00E+00  .25E-04              N  .00E+00  .25E-04
EL HGT            1297.8660 M Z:  4267460.3840  Z  .00E+00  .00E+00  .25E-04     U  .00E+00  .00E+00  .25E-04
  102     TRIMBLE                                X         Y         Z               E         N         U
LAT (N+S-)   42 15 22.596449 X: -2490854.5010  X  .15E-02                       E  .25E-04
LON (E+W-) -121 47  6.118987 Y: -4019681.2420  Y  .24E-02  .40E-02              N  .00E+00  .25E-04
EL HGT            1302.3653 M Z:  4267591.4060  Z -.26E-02 -.42E-02  .45E-02     U  .00E+00  .00E+00  .10E-01
  103     MEDIAN-2                               X         Y         Z               E         N         U
LAT (N+S-)   42 15 15.610080 X: -2491313.1630  X  .10E-01                       E  .10E-01
LON (E+W-) -121 47 25.985930 Y: -4019556.6820  Y  .00E+00  .10E-01              N  .00E+00  .10E-01
EL HGT            1289.8706 M Z:  4267423.4200  Z  .00E+00  .00E+00  .10E-01     U  .00E+00  .00E+00  .10E-01
  104     PUB                                    X         Y         Z               E         N         U
LAT (N+S-)   42 15 32.913540 X: -2490534.8630  X  .15E-02                       E  .25E-04
LON (E+W-) -121 46 54.796867 Y: -4019658.1960  Y  .24E-02  .40E-02              N  .00E+00  .25E-04
EL HGT            1337.7200 M Z:  4267850.8380  Z -.26E-02 -.42E-02  .45E-02     U  .00E+00  .00E+00  .10E-01

LISTING OF POINTS WITH RESPECT TO MASTER P.O.B:          101  K-785
(ASSUMING POSITION OF P.O.B. IS ERRORLESS)

  NUMBER     EAST  SIGMA      NORTH  SIGMA        UP  SIGMA    STATION
    102   172.933  .005      74.172  .005      4.497  .100    TRIMBLE
    103   -42.666  .100    -381.313  .100     -8.007  .100    MEDIAN-2
    104   491.343  .005     333.731  .005     39.826  .100    PUB

INVERSE BETWEEN POINTS

  101     K-785
X = -2490977.0480  LAT (N+S-)   42 15 16.992928    +/-  .0050 METERS    N
Y = -4019738.1880  LON (E+W-) -121 47  9.354217    +/-  .0050 METERS    E   STANDARD DEVIATIONS
Z =  4267460.3840  EL HGT             1297.8660 M  +/-  .0050 METERS    U

 DELTA X/Y/Z WITH SIGMAS     122.5470M +/-  .040M     56.9460M +/-  .063M    131.0220M +/-  .068M
 DELTA E/N/U WITH SIGMAS      74.1715M +/-  .007M    172.9328M +/-  .007M      4.4965M +/-  .100M
 LOCAL PLANE INV: DIST =     188.1679M +/-  .007M  N AZI. =  23 12 52.82 +/-   7.8 SEC

  102     TRIMBLE
X = -2490854.5010  LAT (N+S-)   42 15 22.596449    +/-  .0050 METERS    N
Y = -4019681.2420  LON (E+W-) -121 47  6.118987    +/-  .0050 METERS    E   STANDARD DEVIATIONS
Z =  4267591.4060  EL HGT             1302.3653 M  +/-  .1000 METERS    U

 DELTA X/Y/Z WITH SIGMAS    -122.5470M +/-  .040M    -56.9460M +/-  .063M   -131.0220M +/-  .068M
 DELTA E/N/U WITH SIGMAS     -74.1733M +/-  .007M   -172.9319M +/-  .007M     -4.5021M +/-  .100M
 LOCAL PLANE INV: DIST =     188.1678M +/-  .007M  N AZI. = 203 12 55.00 +/-   7.8 SEC

  101     K-785
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