LIMITED OMNISCIENCE AND THE BOLZANO–WEIERSTRASS PRINCIPLE

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The constructive study of metric spaces requires at first an examination of each classical proposition for numerical content. In classical mathematics it is a theorem that sequences in a compact space have convergent subsequences, but this is not constructively true. For compact intervals on the real line it has long been known that this theorem is nonconstructive because it implies the *Limited Principle of Omniscience* (LPO); here we show that it is equivalent to LPO. At the same time, we obtain other equivalent forms of LPO which concern arbitrary sequences of positive integers.

We follow the strict constructive approach of Errett Bishop [1], discussions of which are found in [4] and [6]. The lack of numerical content in a classical proposition is shown by relating it to a nonconstructive omniscience principle; the result is called a Brouwerian counterexample. Brouwerian counterexamples are discussed in [1], [2], Section 2 of [3] and in [5].

DEFINITIONS. Because the various conditions concerning sequences each have several classical forms, which are constructively different, it is necessary to give explicit constructive definitions. For example, a sequence $\{p_n\}_{n=1}^{\infty}$ of positive integers is *bounded* if a suitable bound has been constructed. Classically, *unbounded* may be understood to mean that the existence of a bound is contradictory, but for a constructive study we adopt rather an affirmative meaning: a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ has been constructed with $p_{n_k} > k$ for all k. Other concepts are similarly interpreted in an affirmative manner. A *decision sequence* is a nondecreasing sequence of 0s and 1s.

THEOREM. *The following are equivalent.*

(a) LPO. Limited Principle of Omniscience. *For any decision sequence* $\{a_n\}$, *either all* $a_n = 0$ *or some* $a_n = 1$.

(b) BSP. Bounded Sequence Principle. *Any sequence of positive integers is either bounded or unbounded.*

(c) KSP. Constant Subsequence Principle. *Any bounded sequence of positive integers has a constant subsequence.*

(d) CSP. Convergent Subsequence Principle. *Any sequence in* $\mathbb{N}$ *has a subsequence convergent in* $\mathbb{N}^* \equiv \mathbb{N} \cup \{\infty\}$.

(e) BWP. Bolzano–Weierstrass Principle. *Any bounded sequence of real numbers has a convergent subsequence.*

(f) MSP. Monotone Sequence Principle. *Any bounded monotone sequence of real numbers converges.*

Proof. (a) implies (b). Given a sequence $\{p_n\}$, define, for all m and n,

$$a_n^m \equiv \bigvee_{k=1}^{n} [0 \vee (p_k - m) \wedge 1].$$

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For each m, $\{a_n^m\}_{n=1}^{\infty}$ is a decision sequence. If $a_n^m = 0$ for all n, define $c_m \equiv 1$, while if $a_n^m = 1$ for some n, define $c_m \equiv 0$. Then $\{c_m\}$ is a decision sequence. If all $c_m = 0$, then $\{p_n\}$ is unbounded, while if there exists m such that $c_m = 1$, then $\{p_n\}$ is bounded by m.

(b) implies (c). If $\{p_n\}$ is bounded by M, then for $1 \leqslant m \leqslant M$ and all n, define $t_n^m \equiv 1$ if $p_n \neq m$ and $t_n^m \equiv n$ if $p_n = m$. For each m with $1 \leqslant m \leqslant M$, the sequence $\{t_n^m\}_{n=1}^{\infty}$ is either bounded or unbounded. If each of these is bounded, choose $j > t_n^m$ for all m and n; then $p_j \neq m$ for all m, a contradiction. Hence there exists m such that the sequence $\{t_n^m\}_{n=1}^{\infty}$ is unbounded, and thus has a subsequence $\{t_{n_k}^m\}_{k=1}^{\infty}$ with all terms greater than 1. Then $\{p_{n_k}\}_{k=1}^{\infty}$ has the constant value m.

(c) implies (e). In the usual proof by interval-halving, there are two nonconstructive steps. The first is easily repaired. Constructively, a closed interval is not the union of two closed half-subintervals. However, by the constructive dichotomy lemma ([**1**, Corollary, p. 24], [**2**, Chapter 2, 2.17], or [**5**]), slightly overlapping intervals may be used. The second, wherein lies the nonconstructivity of BWP, is the decision that, of the terms of a sequence, infinitely many must lie in one of the subintervals. Given a sequence $\{x_n\}$ in an interval, decide, for each n, whether x_n lies in the left or right subinterval, and define $p_n \equiv 1$ or $p_n \equiv 2$ accordingly. Now KSP provides a constant subsequence of $\{p_n\}$, and thus an infinite subsequence of $\{x_n\}$ which lies in one subinterval.

(e) implies (f), and (f) implies (a), are immediate.

(d) is equivalent to (c). It is clear that (d) implies (c). Given (c), let $\{p_n\}$ be a sequence in the set $\mathbb{N}$ of positive integers. Since KSP clearly implies LPO, and hence BSP, it follows that $\{p_n\}$ is either bounded or unbounded. In the first case, KSP yields a convergent subsequence. Thus in either case $\{p_n\}$ has a subsequence convergent in $\mathbb{N}^*$.

REMARK. Brouwerian counterexamples often demonstrate the nonconstructivity of classical theorems by showing that they imply LPO, which in turn implies solutions to a great number of unsolved problems in number theory and analysis; Fermat's Last Theorem and the Riemann Hypothesis are often used as typical examples. However, LPO has not been known to imply solutions to a certain class of unsolved problems, including the Twin Prime Problem, which lead naturally to the question of whether or not a certain sequence of 0s and 1s contains infinitely many 1s. This question is now resolved by the Bounded Sequence Principle: replace each 0 by 1, and each 1, if in the nth place, by n.

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