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CONSTRUCTIVE IRRATIONAL SPACE

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The Fréchet combination allows the construction of a complete metric on the set of irrational numbers. The constructive study of the resulting space $\mathbb{M}$ was begun by Errett Bishop. This paper studies the structure of $\mathbb{M}$ in some detail. The constructive approach requires a strong form of the concept of irrational number and particular attention to the distinctions between the various notions of points exterior to a set. The main results are the characterization and construction of all compact and locally compact subspaces of $\mathbb{M}$.

1. Introduction. An *irrational number* is a real number $x \in \mathbb{R}$ such that $|x-q| > 0$ for every rational number $q \in \mathbb{Q}$. The classically equivalent definition, "x is not in $\mathbb{Q}$", is much weaker when interpreted in a strictly constructive manner; it is insufficient for the formation of the reciprocal $1/|x-q|$ which is required here. We choose a fixed indexing $\mathbb{Q} = \{q_k\}_{k=1}^{\infty}$ of the rationals.

Definition. For any points x and y in $\mathbb{M}$, define

$$d(x,y) \equiv |x-y| + \sum_{k=1}^{\infty} \left| \frac{1}{|x-q_k|} - \frac{1}{|y-q_k|} \right| \wedge \frac{1}{2^k}$$

This yields a metric on $\mathbb{M}$; the effect is to enlarge distances in sequences of irrationals which converge in $\mathbb{R}$ to a rational number, so that in the metric d they are no longer Cauchy sequences. The space $(\mathbb{M}, d)$ is discussed classically in [10, p.173].

The constructive properties of real numbers and metric spaces are found in [1] or [3]. General discussions of the constructive methods introduced by Bishop may be found in [7] and [9].

As for all constructive mathematics, the results obtained here are also classically valid. Certain restrictions required here, however, are not necessary classically. The most notable instance of this is the notion of *located* set, to which distances from any point in a space may be constructively measured; classically, every set is located.

2. Basic properties of $\mathbb{M}$. The basic computational tools needed for the study of $\mathbb{M}$ are contained in Lemmata 2.1 and 2.3 below; the proofs are elementary. The usual metric on $\mathbb{R}$ will be denoted ρ; reference to $\mathbb{R}$ will imply this metric. Reference to $\mathbb{M}$ will imply the metric d, except when the metric ρ is explicitly mentioned. For any subset X of $\mathbb{M}$, the closure of X in $\mathbb{R}$ will be denoted by $\overline{X}$.

LEMMA 2.1. Let $0 < \varepsilon < 1$, let $1/2^N < \varepsilon/3$, and let $0 < \lambda < 1$. If x and y are points of $\mathbb{M}$ such that $|x - q_k| > \lambda$ for $k \leqslant N$ and $|x - y| < \lambda^2\varepsilon/3N$, then $d(x, y) < \varepsilon$.

COROLLARY 2.2. The following are the same with respect to either metric, ρ or d: The open subsets of $\mathbb{M}$. The pointwise continuous real-valued functions on $\mathbb{M}$. The inequality relations on $\mathbb{M}$. Sequential convergence in $\mathbb{M}$. In addition, for any $X \subset \mathbb{M}$, the d-closure of X in $\mathbb{M}$ is $\overline{X} \cap \mathbb{M}$.

LEMMA 2.3. Let $x, y \in \mathbb{M}$. If $|y - q_k| \leqslant |x - q_k|/2$ and $|y - q_k| \leqslant 1$, then $d(x, y) > 1/2^k$.

When $X \subset \mathbb{R}$, and X is located, $-X$ denotes the *metric complement* $\{y \in \mathbb{R} : \rho(y, X) > 0\}$ of X. The notation $-X$ and the term *metric complement* will also be used for arbitrary sets; $-X$ will consist of all points y in $\mathbb{R}$ such that there exists $\lambda > 0$ with $|y - x| \geqslant \lambda$ for all $x \in X$.

Metric equivalence for two metrics on a set will mean uniform continuity of both the identity mapping and its inverse, rather than pointwise continuity, which is used classically.

THEOREM 2.4. Let $X \subset \mathbb{M}$. If $\mathbb{Q} \subset -X$, then ρ and d are equivalent on X.

Proof. Since $\rho < d$ it suffices to show that for any $0 < \varepsilon < 1$ there exists $\delta > 0$ such that whenever $x, y \in X$ and $|x - y| < \delta$, then $d(x, y) < \varepsilon$. Choose N so that $1/2^N < \varepsilon/3$, construct $0 < \lambda < 1$ so that $|x - q_k| > \lambda$ for all $x \in X$ and all $k \leqslant N$, and define $\delta \equiv \lambda^2\varepsilon/3N$.

Examples. To illustrate this result, consider simple subsets of $\mathbb{M}$ of the form S_x: the set of terms of a sequence of irrationals which ρ-converges to a point x in $\mathbb{R}$. For $q \in \mathbb{Q}$, Lemma 2.3 shows that ρ and d are not equivalent on S_q. On the other hand, when $z \in \mathbb{M}$, then $\mathbb{Q} \subset -S_z$, so Theorem 2.4 shows that ρ and d *are* equivalent on S_z. These examples will also illustrate many of the following results. Note that $\overline{S_z} \subset \mathbb{M}$, although constructively $\overline{S_z}$ is not merely $S_z \cup \{z\}$.

Although Corollary 2.2 shows that ρ and d are locally equivalent on $\mathbb{M}$, the example S_q shows that they are not equivalent (in the strong sense); we will try to determine the subspaces of $\mathbb{M}$ on which they are equivalent.

THEOREM 2.5. Fix a point x in $\mathbb{M}$. Define the function $d_x : \mathbb{M} \to \mathbb{R}^{0+}$ from $\mathbb{M}$ to the nonnegative reals by

$$d_x(y) \equiv d(x, y) \qquad (y \in \mathbb{M})$$

Then d_x is ρ-uniformly-continuous on $\mathbb{M}$.

Proof. Since the convergence of the series defining d is uniform, the ρ-uniform-continuity of d_x will follow from that of each term. The k^{th} term is ρ-uniformly-continuous on any interval bounded away from q_k, and constant in some neighborhood of q_k.

Remark. On the other hand, the identity function from $(\mathbb{M}, \rho)$ to $\mathbb{M}$ is not uniformly continuous; that is, the metrics ρ and d are not equivalent on $\mathbb{M}$. This distinction is reflected in the inequality

$$|d(x, y_1) - d(x, y_2)| \leqslant d(y_1, y_2)$$

The metric complement $-X$ of a subset of $\mathbb{R}$ is distinquished from the *negation complement* $X^* \equiv \{y \in \mathbb{R} : y \notin X\}$. Between these is the *strong complement* $X^\# \equiv \{y \in \mathbb{R} : |y - x| > 0 \text{ for all } x \in X\}$. Note that $\mathbb{M} = \mathbb{Q}^\#$. However, Brouwerian counterexamples may be given to show that the following classical statements are constructively invalid: $\mathbb{M} = \mathbb{Q}^*$, $\mathbb{Q} = \mathbb{M}^\#$, $\mathbb{Q} \cup \mathbb{M} = \mathbb{R}$.

We will often need the following lemma due to Bishop; for a proof see [6, 5.4].

LEMMA 2.6. If X is a complete located subset of a metric space Y, then $X^\# = -X$.

3. Subspaces of $\mathbb{M}$. In this section we derive the basic connections between the metrics ρ and d, in regard to totally bounded, compact, and locally compact subspaces of $\mathbb{M}$. In choosing the definition of a concept for constructive development, one of several classically equivalent conditions is carefully chosen for its constructive usefulness; it is usually not constructively equivalent to the other conditions. A metric space is *totally bounded* if it contains finite ε-approximations for every $\varepsilon > 0$, it is *compact* if it is complete and totally bounded, and it is *locally compact* if every bounded subset is contained in a compact subset. Although almost all constructive definitions are chosen so that the results read true classically, an exception is this definition of locally compact space. The classical definition does not serve well in a constructive development; this is discussed further in [1] and [3].

THEOREM 3.1. Let $X \subset \mathbb{M}$. Then the following are equivalent.

(a) X is d-totally-bounded.

(b) X is ρ-totally-bounded and $\mathbb{Q} \subset -X$.

(c) X is ρ-totally-bounded and $\overline{X} \subset \mathbb{M}$.

(d) $\overline{X} \subset \mathbb{M}$ and $\overline{X}$ is d-compact.

Proof. (a) implies (b). Since $\rho < d$, it is clear that X is ρ-totally-bounded. To show $\mathbb{Q} \subset -X$, let $q \in \mathbb{Q}$, choose k so that $q = q_k$, construct a finite $1/2^k$ d-approximation A to X, and define $\lambda \equiv 1 \wedge \min\{|x - q| : x \in A\}$. Now let $y \in X$, and suppose $|y - q| < \lambda/2$. By Lemma 2.3, $d(x, y) > 1/2^k$ for every $x \in A$, contradicting the construction of A; hence $|y - q| \geqslant \lambda/2$.

(b) implies (c). Since $\mathbb{Q} \subset -X = -\overline{X}$, we have $\overline{X} \subset \mathbb{M}$.

(c) implies (d). Since $\overline{X}$ is ρ-complete and located, and $\mathbb{Q} \subset \mathbb{M}^\# \subset \overline{X}^\#$, it follows that $\mathbb{Q} \subset -\overline{X}$. Thus the metrics are equivalent on $\overline{X}$, and since $\overline{X}$ is ρ-compact, it is also d-compact.

(d) implies (a). The d-closure of X is $\overline{X}$, which is d-totally-bounded; thus X is also d-totally-bounded.

COROLLARY 3.2. $\mathbb{M}$ is complete in the metric d.

Proof. Let X be the set of terms of a d-Cauchy sequence in $\mathbb{M}$.

It follows from the last theorem that if $\overline{X}$ meets $\mathbb{Q}$, then X is not d-totally-bounded; this negativistic comment has the following affirmative form. The subset S_q is an example.

PROPOSITION 3.3. Let $X \subset \mathbb{M}$. If $\overline{X}$ meets $\mathbb{Q}$, then X is not d-totally-bounded, in the strong sense that there exists $\varepsilon > 0$ such that for any finitely enumerable subset A of X, there exists $x \in X$ such that $d(x, y) > \varepsilon$ for all $y \in A$.

Proof. Construct $q \in \overline{X} \cap \mathbb{Q}$ and choose k so that $q = q_k$. Let A be any finitely enumerable subset of X, and construct $x \in X$ so that $|x - q| \leqslant |y - q|/2$ for all $y \in A$, and $|x - q| \leqslant 1$. Then for any $y \in A$ it follows from Lemma 2.3 that $d(x, y) > 1/2^k$.

Let $\{x_n\}$ be a sequence in $\mathbb{M}$, convergent with respect to ρ to a rational point. Because $\mathbb{M}$ is complete, the sequence cannot be d-Cauchy (Corollary 2.2). This negativistic comment has the following affirmative form, a slightly strengthened version of a statement in [1, p. 110]; the proof is similar to that of the last proposition.

PROPOSITION 3.4. A sequence $\{x_n\}$ in $\mathbb{M}$, convergent with respect to ρ to a rational point, is not d-Cauchy, in the strong sense that there exists $\varepsilon > 0$ such that for any n there exists $m > n$ such that $d(x_m, x_n) > \varepsilon$.

THEOREM 3.5. Let $X \subset \mathbb{M}$. Then X is compact with respect to d if and only if it is compact with respect to ρ. In this situation, $\mathbb{Q} \subset -X$ and the metrics ρ and d are equivalent on X.

Proof. Theorems 3.1 and 2.4.

THEOREM 3.6. Let $X \subset \mathbb{M}$. Then X is locally compact with respect to d if and only if it is locally compact with respect to ρ. In this situation, $\mathbb{Q} \subset -X$ and the metrics ρ and d are equivalent on X. [1, p.109]

Proof. The bounded subsets of X are clearly the same with respect to either metric. By the last theorem, the compact subsets are the same. When X is locally compact with respect to ρ, then X is a closed located subset of $\mathbb{R}$; thus $\mathbb{Q} \subset -X$, and the metrics are equivalent on X.

4. Characterization of locally compact and compact subspaces. A subspace of a locally compact space is locally compact [compact] if and only if it is closed and located [and bounded]. Although the space $\mathbb{M}$ is not locally compact, we obtain similar characterizations for subspaces of $\mathbb{M}$, together with a method for the construction of all such subspaces.

THEOREM 4.1. Any subspace T of $\mathbb{M}$ such that $\mathbb{Q} \subset -T$ can be enlarged to a subspace X of $\mathbb{M}$ which is locally compact with respect to both ρ and d. If, in addition, T is bounded, then X can be constructed so as to be compact with respect to both ρ and d.

Proof. We utilize the method of notches given in [5]. Construct a sequence $\{\lambda_k\}$ so that $\lambda_k \searrow 0$, and $|q_k - x| \geq \lambda_k$ for all k and all $x \in T$. Define $k_1 \equiv 1$, construct a positive irrational $\alpha_1 \leq \lambda_1$, and define $I_1 \equiv S(q_{k_1}, \alpha_1)$, the open interval in $\mathbb{R}$ about q_{k_1} of radius α_1. Now let $n > 1$ and suppose that for all $i < n$, integers k_i have been chosen, positive irrationals α_i have been constructed, and intervals $I_i \equiv S(q_{k_i}, \alpha_i)$ have been defined, so that k_i is the first integer with $q_{k_i} \in -\bigcup_{j<i} I_j$, and $\alpha_i \leq \lambda_{k_i} \wedge \rho(q_{k_i}, \bigcup_{j<i} I_j)$. Since the endpoints of the intervals I_i are irrational, it follows that for each k either $q_k \in \bigcup_{i<n} I_i$ or $q_k \in -\bigcup_{i<n} I_i$. Thus k_n and α_n may be constructed to continue the induction.

This defines a sequence $\{I_n\}_{n=1}^{\infty}$ of disjoint open intervals with lengths $\ell(I_n) \to 0$, and yields a closed located subset $X \equiv \bigcap_{n=1}^{\infty} I_n^*$ of $\mathbb{R}$ with $-X = \bigcup_{n=1}^{\infty} I_n$. Thus X is ρ-locally-compact. It follows from the construction that $T \subset X \subset \mathbb{M}$; thus X is d-locally-compact.

When T is bounded, there is a compact subset Y of X which contains T. (Alternatively, the construction of X may be modified so as to also notch out the intervals $(-\infty, a)$ and $(b, +\infty)$, where $T \subset [a, b]$, and a and b are irrational. The resulting set X is then compact.)

The construction of [5] yields all closed located subsets of $\mathbb{R}$. By adding the condition that all rational points are notched out, all locally compact (and compact) subsets of $\mathbb{M}$ are obtained.

THEOREM 4.2. A subspace X of $\mathbb{M}$ is locally compact if and only if it is closed and located, and $\mathbb{Q} \subset -X$.

Proof. The necessity follows from Theorem 3.6. Conversely, enlarge X to a locally compact subspace Y; since X is closed and located in Y, it is locally compact.

THEOREM 4.3. A subspace X of $\mathbb{M}$ is compact if and only if it is closed, located, and bounded, and $\mathbb{Q} \subset -X$.

Proof. The necessity follows from Theorem 3.5. Conversely, the preceding result shows that X is locally compact; since it is now bounded, it is compact. (Alternatively, apply Theorem 4.1 directly.)

Remark. If the condition "bounded" is viewed as "bounded away from infinity", then the condition "$\mathbb{Q} \subset -X$", that is, "bounded away from each rational point", is seen as a direct extension. Under the metric d, each rational point appears, in a sense, as an infinity. The condition for a compact set is that it be bounded away from all infinities; for a locally compact set, from all but one.

5. Equivalent subspaces. The following theorem, which continues Theorem 2.4, gives one sufficient, and one necessary, condition for the metrics ρ and d to be equivalent on a given subspace X of $\mathbb{M}$. Examples below show that the one condition is *not* necessary, while the other is *not* sufficient. Thus it remains an open problem to give a simple necessary and sufficient topological condition for metric equivalence.

THEOREM 5.1. Let $X \subset \mathbb{M}$. Then each of the following conditions implies the next.

(a) $\mathbb{Q} \subset -X$
(b) The metrics ρ and d are equivalent on X.
(c) $\overline{X} \subset \mathbb{M}$

Proof. Given (b), the set S of terms of a sequence in X, which is ρ-convergent to a point x of $\overline{X}$, is not only ρ-totally-bounded, but also d-totally-bounded. By Theorem 3.1, $\overline{S} \subset \mathbb{M}$.

COROLLARY 5.2. Let $X \subset \mathbb{M}$ and let X be located in $\mathbb{R}$. Then the conditions of the preceding theorem are equivalent.

Proof. If X is located, then $\overline{X}$ is locally compact, and if also $\overline{X} \subset \mathbb{M}$, then it follows from Theorem 3.6 that $\mathbb{Q} \subset -\overline{X} = -X$.

It follows from Theorem 5.1 that if $\overline{X}$ meets $\mathbb{Q}$, then ρ and d are not equivalent on X; this negativistic comment has the following affirmative form, which follows from Lemma 2.3.

PROPOSITION 5.3. Let $X \subset \mathbb{M}$. If $\overline{X}$ meets $\mathbb{Q}$, then ρ and d are not equivalent on X, in the strong sense that there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exist $x, y \in X$ such that $|x-y| < \delta$ but $d(x, y) > \varepsilon$.

Since classically all subsets of $\mathbb{M}$ are located, Brouwerian counterexamples are required to show the nonconstructivity of the converses to Theorem 5.1. A *decision sequence* is a nondecreasing sequence $\{a_n\}$ of 0's and 1's. There is no general constructive procedure which determines, for an arbitrary decision sequence, whether or not there exists a term equal to 1. The *Limited Principle of Omniscience* (LPO), applied to any decision sequence, either predicts that all the terms are 0, or indicates the location of a 1. This principle is nonconstructive, and therefore so is any statement which implies it. The technique of Brouwerian counterexamples is described more fully in [1], [2], section 2 of [6], and [8].

EXAMPLE 5.4. The statement "If X is a nonvoid subspace of $\mathbb{M}$ on which the metrics ρ and d are equivalent, then $\mathbb{Q} \subset -X$" is nonconstructive; it implies LPO.

Proof. Given any decision sequence $\{a_n\}$, define

$$X \equiv \{\sqrt{3}\} \cup \{\sqrt{2}/n : a_n < a_{n+1}\}$$

If $x, y \in X$ with $|x - y| < 1/4$, then $x = y$, so ρ and d are equivalent on X. By hypothesis, X is bounded away from 0, so $\{a_n\}$ is eventually constant.

EXAMPLE 5.5. The statement "If X is a nonvoid subspace of $\mathbb{M}$ with $\overline{X} \subset \mathbb{M}$, then the metrics ρ and d are equivalent on X" is nonconstructive; it implies LPO.

Proof. Given any decision sequence $\{a_n\}$, define

$$X \equiv \{\sqrt{3}\} \cup \{\sqrt{2}/n : a_n < a_{n+1}\} \cup \{\sqrt{2}/2n : a_n < a_{n+1}\}$$

Let $\{x_n\}$ be a sequence in X which is ρ-convergent to a point x in $\mathbb{R}$. Eventually, $|x_i - x_j| < 1/4$; thus either $x_n = \sqrt{3}$ eventually, so $x \in X$, or $a_n = 1$ for some n, in which event X is closed and again $x \in X$. Thus X is closed in $\mathbb{R}$, so $\overline{X} \subset \mathbb{M}$. Choose k so that $q_k = 0$. By hypothesis, construct $\delta > 0$ so that $d(x, y) < 1/2^k$ whenever $|x - y| < \delta$, and choose $N > \sqrt{2}/2\delta$. If $a_N = 0$, then it follows from Lemma 2.3 that all $a_n = 0$.

6. Located subsets. We now compare located subsets of $\mathbb{M}$ and $(\mathbb{M}, \rho)$. Recall that a subspace of $\mathbb{R}$ is totally bounded if and only if it is bounded and located in $\mathbb{R}$. For the first result we need the following lemma, a slight generalization of part of Proposition 13 in Chap. 4 of [1].

LEMMA 6.1. Let Y be a metric space and F a nonvoid subset of Y. If every bounded subset of F is contained in a subset of F which is located in Y, then F is located in Y.

THEOREM 6.2. Let $X \subset \mathbb{M}$. If X is located in $\mathbb{R}$, then X is also located in $\mathbb{M}$.

Proof. First consider the special case in which X is bounded, and let $x \in \mathbb{M}$. Since the function $y \to d(x, y)$ is ρ-uniformly-continuous on $\mathbb{M}$, and X is ρ-totally-bounded, $d(x, X) \equiv \inf\{d(x, y) : y \in X\}$ exists. Hence X is located in $\mathbb{M}$.

Now consider the general case, and let B be any bounded subset of X. It follows from Theorem 1 of [4] that there exist $a, b \in \mathbb{R}$ such that $X \cap [a, b]$ is located in $\mathbb{R}$ and contains B. By the first part of the proof, $X \cap [a, b]$ is located in $\mathbb{M}$.

The converse is true in the special situation covered by the next proposition, but whether it is true in general is an open problem.

PROPOSITION 6.3. Let $X \subset \mathbb{M}$ with $\mathbb{Q} \subset -X$. Then X is located in $\mathbb{M}$ if and only if it is located in $\mathbb{R}$.

Proof. Let X be located in $\mathbb{M}$. Since $\overline{X}$ is contained in $\mathbb{M}$, it is the closure of X in $\mathbb{M}$, and thus it is located in $\mathbb{M}$. It follows from Theorem 4.2 that $\overline{X}$ is d-locally-compact, hence ρ-locally-compact, and thus located in $\mathbb{R}$.

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