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Constructive irrational space.
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Let M be the set of irrational numbers. Let q_k enumerate the set $\mathbf{Q}$ of rational numbers. A metric d on M is defined by

$$d(x,y)=|x-y|+\sum_{k=1}^{\infty}2^{-k}||x-q_k|^{-1}-|y-q_k|^{-1}|.$$

The point of this metric is that it makes each rational a "point at infinity", similar to the usual infinity. The relations between this metric and the metric ρ on M induced by the usual metric on the reals are the object of study in this paper. While these questions are mathematically interesting in their own right, it should be noted that the paper is not being reviewed in the classification appropriate to such mathematical questions, but in the section on constructive mathematics. It is a sign of the climate of the times that methods take precedence over subject here. The questions considered in this paper concern the existence or nonexistence of certain algorithms. Since these algorithms involve abstract mathematical objects (sets of irrational numbers), the framework of Bishop's constructive mathematics is appropriate for their statement and solution, rather than the explicit discussion of algorithms manipulating representations of the abstract objects.

The point of constructive mathematics is that it permits a shorthand style of discourse which looks much like normal abstract mathematics, yet contains implicitly a fully precise description of algorithms to carry out constructions. For example, when one says "X is a compact set" in constructive mathematics, this is short for assuming that one has an algorithm for constructing from input n an ε-net of points $x_1,\cdots,x_n$ which approximates each point of X to within ε, and that X is complete (Cauchy sequences converge to points of X). Every statement that one makes in constructive mathematics can be expanded on demand to a statement in which every "there exists", even those which are implicit in the definitions of the concepts used, refers to an algorithm. The value of this systematic use of definitions is really a natural extension of the way in which definitions are always used in mathematics. The paper under review furnishes an excellent example of this sort of thing.

For example, a subset X of some metric space M is said to be located if we have an algorithm for computing the distance from a given point of M to X. Note that this statement admits of several interpretations, depending on the exact choice of representation of the infinite objects these algorithms must work on; the beauty of systematic constructive mathematics lies in the fact that one need not worry about such details, and the results are independent of such things as the choice of representation of irrational numbers and the definition of "algorithm".

In the paper under review, a typical question considered is this: suppose the subset X of M is located using one of the two metrics d and ρ. Is it then necessarily located using the other metric? It is proved that if X is ρ-located, then it is d-located. The converse is proved under an additional assumption, viz. that $\mathbf{Q}$ is contained in the "metric complement" of X. Translated into the language of algorithms, that means that we have an algorithm φ which takes a rational q as input and produces a positive lower bound $\varphi(q) = 1/m$ on the distance from q to X. Note that in general subsets of M are not contained in the metric complement of $\mathbf{Q}$; although we cannot give a counterexample, we also have no way of proceeding algorithmically from (the definition of) a subset X of M to an algorithm φ. Implicit in the proof of these theorems is a method of transforming a ρ-distance-to-X algorithm to a d-distance-to-X algorithm, and conversely, of transforming a d-distance-to-X algorithm together with a φ algorithm for X into a ρ-distance-to-X algorithm. It remains an open question whether φ is really needed to perform this latter kind of transformation.

The point of the paper, aside from the direct mathematical point, is this: if one were to try to obtain these algorithmic results without the aid of the systematic definitions and theorems of constructive mathematics, the welter of detail would be overwhelming. As it is, the argument requires only a few pages, and is clean, direct and comprehensible.

Reviewed by *Michael Beeson*