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The common point problem in constructive projective geometry

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Abstract

Using intuitionistic methods, an extension of an incidence plane was constructed by Heyting in 1959; however, a central question, the validity of the projective axiom that any two lines have a common point, was left open. A Brouwerian counterexample demonstrates that in the Heyting extension the common point axiom is constructively invalid.
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0. Introduction

An extension of an incidence plane has been constructed by Heyting [6], using intuitionistic methods [7], although the validity of the projective axiom that any two lines have a common point was not established. Work by van Dalen [5] developed the subject further, and improved the axiom system; still, the problem of the common point axiom remained open. The Brouwerian counterexample below shows that in the Heyting extension the common point axiom is constructively invalid.[1]

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[1] For an exposition of the constructivist program, see Errett Bishop's "Constructivist Manifesto", Chapter 1 in [1] or [3]; see also [9,13,14]. For a discussion of the philosophical issues motivating a constructive approach to mathematics, see [2].

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A projective extension of an incidence plane, in which the common point axiom is valid, will be constructed in [12].

1. Preliminaries

An incidence plane $(\mathscr{P}, \mathscr{L})$ of points and lines is given, with the basic axioms of [6,5]. The Heyting extension (Π, Λ) of this plane consists of *p.points* of the form

$$\mathfrak{P}(l, m) := \{n \in \mathscr{L} : n \cap l = l \cap m \text{ or } n \cap m = l \cap m\},$$

where $l, m \in \mathscr{L}$ with $l \neq m$, and *p.lines* of the form

$$\lambda(\mathfrak{A}, \mathfrak{B}) := \{\mathfrak{Q} \in \Pi : \mathfrak{Q} \cap \mathfrak{A} = \mathfrak{A} \cap \mathfrak{B} \text{ or } \mathfrak{Q} \cap \mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}\}$$

where $\mathfrak{A}, \mathfrak{B} \in \Pi$ with $\mathfrak{A} \neq \mathfrak{B}$.

For the Heyting extension of the real plane $\mathbb{R}^2$, a simple notation will be used to construct certain p.points. For example, $\mathfrak{X} := \mathfrak{P}(y = 0,\ y = 1)$ is the pencil of horizontal lines; similarly, $\mathfrak{Y}$ is the pencil of vertical lines. The *line at infinity* is $\iota := \lambda(\mathfrak{X}, \mathfrak{Y})$. When the lines l and m intersect, with common point Q, the p.point $\mathfrak{P}(l, m)$ will be denoted Q^*, the pencil of lines through Q.

2. Counterexample to the common point axiom

To determine the specific nonconstructive elements in a classical theory, and thereby to indicate feasible directions for constructive work, *Brouwerian counterexamples* are used, in conjunction with *omniscience principles.* A Brouwerian counterexample is a proof that a given statement implies an omniscience principle. In turn, an omniscience principle would imply solutions or significant information for a large number of well-known unsolved problems.[2] A statement is considered *constructively invalid* if it implies an omniscience principle.[3]

We will need the following omniscience principle.

Lesser limited principle of omniscience (LLPO). *For any real number α, either $\alpha \leq 0$ or $\alpha \geq 0$.*[4]

Brouwerian counterexample. *In the Heyting extension, the statement "Any two p.lines have a common p.point" is constructively invalid; the statement implies LLPO.*

Proof. Let α be any real number; set $\alpha^+ := \max\{\alpha, 0\}$, and $\alpha^- := \max\{-\alpha, 0\}$. In the Heyting extension of the real plane $\mathbb{R}^2$, define

$$\mathfrak{A} := \mathfrak{P}(y = 0,\ y = 1 - \alpha^+ x)$$
$$\mathfrak{B} := \mathfrak{P}(x = 0,\ x = 1 - \alpha^- y)$$
$$\mu := \lambda(\mathfrak{A}, \mathfrak{Y}) \qquad \nu := \lambda(\mathfrak{B}, \mathfrak{X}).$$

By hypothesis, the p.lines μ and ν have a common p.point $\mathfrak{C}$. Using the cotransitivity property for p.points, Theorem 7(iii) in [6], we have either $\mathfrak{C} \neq \mathfrak{X}$ or $\mathfrak{C} \neq \mathfrak{Y}$. In the first case, suppose that

[2] This method was introduced in 1908 by Brouwer [4], to demonstrate that use of the *law of excluded middle* inhibits mathematics from attaining its full significance.

[3] For more information concerning Brouwerian counterexamples, and other omniscience principles, see [1] or [3,8,10].

[4] The omniscience principle LLPO was formulated by E. Bishop [2].

$\alpha < 0$. Then $\alpha^+ = 0$, so $\mathfrak{A} = \mathfrak{X}$, and $\mu = \iota$. Also, $\mathfrak{B} = (0,\ 1/\alpha^-)^*$, so $\mathfrak{B} \notin \mu$. Thus the p.lines μ and ν are distinct, with unique common p.point $\mathfrak{X}$, a contradiction. Hence $\alpha \geq 0$. Similarly, when $\mathfrak{C} \neq \mathfrak{Y}$, we find that $\alpha \leq 0$. Thus LLPO results. □

Note. This counterexample concerns the full common point axiom, rather than the limited Axiom P3 as stated in [6], where only distinct lines are considered. An investigation into the full axiom is necessary for a constructive study based upon numerical meaning, as proposed by Bishop. Questions of distinctness are at the core of constructive problems; any attempted projective extension of the real plane is certain to contain innumerable pairs of lines which may or may not be distinct.

3. Heyting axioms on the real plane

Since Axioms A1 through A7 were used in [6] to establish cotransitivity, verification of these axioms for the real plane is required to support the Brouwerian counterexample above. Only Axiom A1 will require special consideration.

Heyting's Axiom A1. *If l and m are distinct lines, and P is a point outside l, then there exists a line n passing through P such that $n \cap l = m \cap l$.*

Theorem. *On the real plane $\mathbb{R}^2$, the Heyting axioms* A1 *through* A7 *are valid.*

Proof. Since $\mathbb{R}$ is a Heyting field, $\mathbb{R}^2$ satisfies axiom groups **G** and **L** of [11]; this was shown in Section 9 of [11]. Thus the axioms and results in Section 2 of [11] apply here.

(a) *Axiom* A1. We may estimate the angle between the lines l and m. If this angle is positive, the lines will intersect (cf. Lemma 9.7 in [11]), and we can easily draw the required line n. Thus we may assume that the angle is fairly small. Since $P \notin l$, it follows from Theorem 10.1 in [11] that $\rho(P, l) > 0$; set $d := \min\{1, \rho(P, l)\}$. Either $\rho(P, m) > 0$ or $\rho(P, m) < d$.

Case 1. $\rho(P, m) > 0$. Choose distinct points Q, Q' on m, each outside the line l. Since PQ intersects PQ', we may assume, using axiom L2, that PQ intersects l. Choose a coordinate system so that the line l has equation $y = 0$, the line PQ has equation $x = 0$, and the point Q has coordinates (0, 1). Then the line m will have an equation of the form $y = ex + 1$, and the point P will have coordinates of the form $(0, h)$, with $h \neq 0$. Define the line n by the equation $y = hex + h$. It follows that $P \in n$, and it is clear that $n \cap l = m \cap l$.

Case 2. $\rho(P, m) < d$. Choose a point $Q \in m$ so that $\rho(P, Q) < d$; thus $Q \notin l$. Now choose a coordinate system so that the line l has equation $y = 0$, the line $x = 0$ is the perpendicular to l dropped from Q, and the point Q has coordinates (0, 1); this preserves angles. Set $P' := (0, 3)$, then $\rho(P', m) > 0$. Thus Case 1 applies to the configuration (l, m, P'), so we may construct a line m' through P' such that $m' \cap l = m \cap l$. Clearly, $m' \neq l$. Also, since the angle between the lines l and m is small, we have $\rho(P, m') > 0$, so $P \notin m'$. Now Case 1 applies to the configuration (l, m', P), and we may draw a line n through P such that $n \cap l = m' \cap l$. It follows that $n \cap l = m \cap l$.

(b) *Axioms* A2–A7. Using the results of Section 2 in [11], these axioms are easily verified for $\mathbb{R}^2$. □

References

[1] E. Bishop, Foundations of Constructive Analysis, McGraw-Hill Book Co, New York, Toronto, London, 1967.
[2] E. Bishop, Schizophrenia in contemporary mathematics, in: AMS Colloquium Lectures, Missoula, Montana, 1973. Reprinted in Contemp. Math. 39 (1985), pp. 1–32.

[3] E. Bishop, D. Bridges, Constructive Analysis, Springer-Verlag, Berlin, 1985.

[4] L.E.J. Brouwer, De onbetrouwbaarheid der logische principes, Tijdschr. Wijsbeg. 2 (1908) 152–158; English translation A. Heyting (Ed.), The Unreliability of the Logical Principles, in: L.E.J. Brouwer: Collected Works 1: Philosophy and Foundations of Mathematics, Elsevier, Amsterdam, New York, 1975, pp. 107–111.

[5] D. van Dalen, Extension problems in intuitionistic plane projective geometry I, II, Indag. Math. 25 (1963) 349–383.

[6] A. Heyting, Axioms for intuitionistic plane affine geometry, in: L. Henkin, P. Suppes, A. Tarski (Eds.), The Axiomatic Method, with Special Reference to Geometry and Physics: Proceedings of an International Symposium Held at the University of California, Berkeley, December 26, 1957–January 4, 1958, North-Holland, Amsterdam, 1959, pp. 160–173.

[7] A. Heyting, Intuitionism: An Introduction, North-Holland, Amsterdam, 1966.

[8] M. Mandelkern, Constructive continuity, Mem. Amer. Math. Soc. 42 (277) (1983).

[9] M. Mandelkern, Constructive mathematics, Math. Mag. 58 (1985) 272–280.

[10] M. Mandelkern, Brouwerian counterexamples, Math. Mag. 62 (1989) 3–27.

[11] M. Mandelkern, Constructive coordinatization of Desarguesian planes, Beiträge Algebra Geom. 48 (2007) 547–589.

[12] M. Mandelkern, Constructive projective extension of an incidence plane, Trans. Amer. Math. Soc. (in press). Preprint: www.zianet.com/mandelkern/cpe.pdf.

[13] F. Richman, Meaning and information in constructive mathematics, Amer. Math. Monthly 89 (1982) 385–388.

[14] G. Stolzenberg, Review of E. Bishop, Foundations of constructive analysis, Bull. Amer. Math. Soc. 76 (1970) 301–323.