The common point problem in constructive projective geometry

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Abstract
Using intuitionistic methods, an extension of an incidence plane was constructed by Heyting in 1959; however, a central question, the validity of the projective axiom that any two lines have a common point, was left open. A Brouwerian counterexample demonstrates that in the Heyting extension the common point axiom is constructively invalid.

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Introduction
An extension of an incidence plane has been constructed by A. Heyting [H59], using intuitionistic methods [H66], although the validity of the projective axiom that any two lines have a common point was not established. Work by D. van Dalen [D63] developed the subject further, and improved the axiom system; still, the problem of the common point axiom remained open. The Brouwerian counterexample below shows that in the Heyting extension the common point axiom is constructively invalid.1

A projective extension of an incidence plane, in which the common point axiom is valid, will be constructed in [M12].

1 Preliminaries
An incidence plane \((\mathcal{P}, \mathcal{L})\) of points and lines is given, with the basic axioms of [H59] and [D63]. The Heyting extension \((\Pi, A)\) of this plane consists of \emph{p.points}

\footnote{For an exposition of the constructivist program, see Errett Bishop's "Constructivist Manifesto", Chapter 1 in [B67] or [BB85]; see also [M85], [R82], and [S70]. For a discussion of the philosophical issues motivating a constructive approach to mathematics, see [B73].}
of the form
\[ \mathcal{P}(l, m) := \{ n \in \mathcal{L} : n \cap l = l \cap m \text{ or } n \cap m = l \cap m \} \]
where \( l, m \in \mathcal{L} \) with \( l \neq m \), and \( p\text{-}lines \) of the form
\[ \lambda(\mathfrak{A}, \mathfrak{B}) := \{ \Omega \in \Pi : \Omega \cap \mathfrak{A} = \mathfrak{A} \cap \mathfrak{B} \text{ or } \Omega \cap \mathfrak{B} = \mathfrak{A} \cap \mathfrak{B} \} \]
where \( \mathfrak{A}, \mathfrak{B} \in \Pi \) with \( \mathfrak{A} \neq \mathfrak{B} \).

For the Heyting extension of the real plane \( \mathbb{R}^2 \), a simple notation will be used to construct certain \( p\text{-}points \). For example, \( \mathfrak{X} := \mathcal{P}(y = 0, y = 1) \) is the pencil of horizontal lines; similarly, \( \mathfrak{V} \) is the pencil of vertical lines. The \textit{line at infinity} is \( \iota := \lambda(\mathfrak{X}, \mathfrak{V}) \). When the lines \( l \) and \( m \) intersect, with common point \( Q \), the \( p\text{-}point \) \( \mathcal{P}(l, m) \) will be denoted \( Q^* \), the pencil of lines through \( Q \).

## 2 Counterexample to the common point axiom

To determine the specific nonconstructive elements in a classical theory, and thereby to indicate feasible directions for constructive work, \textit{Brouwerian counterexamples} are used, in conjunction with \textit{omniscience principles}. A Brouwerian counterexample is a proof that a given statement implies an omniscience principle. In turn, an omniscience principle would imply solutions or significant information for a large number of well-known unsolved problems.\(^2\) A statement is considered \textit{constructively invalid} if it implies an omniscience principle.\(^3\)

We will need the following omniscience principle:

**Lesser Limited Principle of Omniscience (LLPO).** For any real number \( \alpha \), either \( \alpha \leq 0 \) or \( \alpha \geq 0 \).\(^4\)

**Brouwerian counterexample.** In the Heyting extension, the statement \textit{“Any two \( p\text{-}lines \) have a common \( p\text{-}point \)”} is constructively invalid; the statement implies LLPO.

\begin{proof}
Let \( \alpha \) be any real number; set \( \alpha^+ := \max\{\alpha, 0\} \), and \( \alpha^- := \max\{-\alpha, 0\} \). In the Heyting extension of the real plane \( \mathbb{R}^2 \), define
\[
\mathfrak{A} := \mathcal{P}(y = 0, y = 1 - \alpha^+ x) \\
\mathfrak{B} := \mathcal{P}(x = 0, x = 1 - \alpha^- y)
\]
\end{proof}

\(^2\)This method was introduced in 1908 by L. E. J. Brouwer [Br08], to demonstrate that use of the \textit{law of excluded middle} inhibits mathematics from attaining its full significance.

\(^3\)For more information concerning Brouwerian counterexamples, and other omniscience principles, see [B67] or [BB85], [M83], and [M89].

\(^4\)The omniscience principle LLPO was formulated by Errett Bishop [B73].
$\mu := \lambda(\mathfrak{A}, \mathfrak{Y}) \quad \nu := \lambda(\mathfrak{B}, \mathfrak{X})$

By hypothesis, the p.lines $\mu$ and $\nu$ have a common p.point $\mathcal{C}$. Using the co-transitivity property for p.points, Theorem 7(iii) in [H59], we have either $\mathcal{C} \neq \mathfrak{X}$ or $\mathcal{C} \neq \mathfrak{Y}$. In the first case, suppose that $\alpha < 0$. Then $\alpha^+ = 0$, so $\mathfrak{A} = \mathfrak{X}$, and $\mu = \nu$. Also, $\mathfrak{B} = (0, 1/\alpha^-)^*$, so $\mathfrak{B} \notin \mu$. Thus the p.lines $\mu$ and $\nu$ are distinct, with unique common p.point $\mathfrak{X}$, a contradiction. Hence $\alpha \geq 0$. Similarly, when $\mathcal{C} \neq \mathfrak{Y}$, we find that $\alpha \leq 0$. Thus LLPO results.

**Note.** This counterexample concerns the full common point axiom, rather than the limited Axiom P3 as stated in [H59], where only distinct lines are considered. An investigation into the full axiom is necessary for a constructive study based upon numerical meaning, as proposed by Bishop. Questions of distinctness are at the core of constructive problems; any attempted projective extension of the real plane is certain to contain innumerable pairs of lines which may or may not be distinct.

### 3 Heyting axioms on the real plane

Since Axioms A1 through A7 were used in [H59] to establish cotransitivity, verification of these axioms for the real plane is required to support the Brouwerian counterexample above. Only Axiom A1 will require special consideration.

**Heyting’s Axiom A1.** If $l$ and $m$ are distinct lines, and $P$ is a point outside $l$, then there exists a line $n$ passing through $P$ such that $n \cap l = m \cap l$.

**Theorem.** On the real plane $\mathbb{R}^2$, the Heyting axioms A1 through A7 are valid.

**Proof.** Since $\mathbb{R}$ is a Heyting field, $\mathbb{R}^2$ satisfies axiom groups $G$ and $L$ of [M07]; this was shown in Section 9 of [M07]. Thus the axioms and results in Section 2 of [M07] apply here.

(a) **Axiom A1.** We may estimate the angle between the lines $l$ and $m$. If this angle is positive, the lines will intersect (cf. Lemma 9.7 in [M07]), and we can easily draw the required line $n$. Thus we may assume that the angle is fairly small. Since $P \notin l$, it follows from Theorem 10.1 in [M07] that $\rho(P, l) > 0$; set $d := \min\{1, \rho(P, l)\}$. Either $\rho(P, m) > 0$ or $\rho(P, m) < d$.

**Case 1.** $\rho(P, m) > d$. Choose distinct points $Q, Q'$ on $m$, each outside the line $l$. Since $PQ$ intersects $PQ'$, we may assume, using axiom L2, that $PQ$ intersects $l$. Choose a coordinate system so that the line $l$ has equation $y = 0$, the line $PQ$ has equation $x = 0$, and the point $Q$ has coordinates $(0, 1)$. Then the line $m$ will have an equation of the form $y = ex + 1$, and the point $P$ will have coordinates of the form $(0, h)$, with $h \neq 0$. Define the line $n$ by the equation $y = hex + h$. It follows that $P \in n$, and it is clear that $n \cap l = m \cap l$.

**Case 2.** $\rho(P, m) < d$. Choose a point $Q \in m$ so that $\rho(P, Q) < d$; thus $Q \notin l$. Now choose a coordinate system so that the line $l$ has equation $y = 0$, ...
the line $x = 0$ is the perpendicular to $l$ dropped from $Q$, and the point $Q$ has coordinates $(0,1)$; this preserves angles. Set $P' := (0,3)$, then $\rho(P', m) > 0$. Thus Case 1 applies to the configuration $(l, m, P')$, so we may construct a line $m'$ through $P'$ such that $m' \cap l = m \cap l$. Clearly, $m' \neq l$. Also, since the angle between the lines $l$ and $m$ is small, we have $\rho(P, m') > 0$, so $P \notin m'$. Now Case 1 applies to the configuration $(l, m', P)$, and we may draw a line $n$ through $P$ such that $n \cap l = m' \cap l$. It follows that $n \cap l = m \cap l$.

(b) Axioms A2-A7. Using the results of Section 2 in [M07], these axioms are easily verified for $\mathbb{R}^2$. □

References


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