Constructive Mathematics

Recent advances in constructive mathematics draw attention to the importance of proofs with numerical meaning.

MARK MANDELKERN
2315 Derby Street
Berkeley, CA 94705

An age-old controversy in mathematics concerns the necessity and the possibility of constructive proofs. While moribund for almost fifty years, the controversy has been rekindled by recent advances which demonstrate the feasibility of a fully constructive mathematics. This nontechnical article discusses the motivating ideas behind this approach to mathematics and the implications of constructive mathematics for the history of mathematics.

For over a hundred years the controversy over constructivity has been simmering silently beneath the surface of mathematics, occasionally erupting into full-blown battle, but never reaching a settlement. Still the conflict continues. Today, while the vast majority in mathematics use nonconstructive methods, a small minority persist in the struggle to bring constructive methods into general use.

During a previous episode in this controversy, Einstein asked, "What is this frog-and-mouse battle among the mathematicians?" [**20**, p. 187]. One might ask the same today. To begin with a brief answer which is elaborated below, constructive proofs are those which ultimately reduce to finite constructions with the integers $1, 2, 3, \ldots$. Such proofs are said to have *numerical meaning.* In *constructive* mathematics, numerical meaning is central. In contrast, *classical* mathematics, which is dominant today, admits methods which are not in essence finite, with the result that classical proofs often lack numerical meaning.

Recent advances in constructivity, which point to a final resolution of the problem, stem from Errett Bishop's 1967 book, *Foundations of Constructive Analysis* [**1**]. A growing number of mathematicians now work in accord with the general principles proposed and developed by Bishop.

The present article discusses the basic ideas which motivate this group of modern constructivists. The treatment deals with no technical details, but rather with fundamental human attitudes towards mathematics, its meaning, and its purpose. (For a thorough technical discussion, see [**25**].) The author is indebted to Y. K. Chan, Michael Goldhaber, Keith Phillips, and the late Errett Bishop for critical readings of early drafts of this article.

Historical background

The present controversy in mathematics has traces even in the early history of mathematics.

> *Not geometry, but arithmetic alone will provide satisfactory proof.*
> Archytas, ca. 375 B.C. [**13**, p. 49]

Archytas refers to an ancient controversy concerning geometric and arithmetic proofs. Certain aspects of this controversy correspond roughly to the present controversy between classical and constructive proofs.

Throughout the history of mathematics, both constructive and nonconstructive tendencies are found. The Pythagoreans tried to reduce all mathematics to numbers. Plato introduced, through his theory of forms, an idealistic approach to philosophical problems which pervades all classical mathematics; he taught that truth exists independently of humans, who must seek it through dialectic. Aristotle began the systemization of logic; his *principle of excluded middle* (discussed below) is a major cause of nonconstructivities when applied to the mathematical infinite. Gauss first gave his complex numbers a geometric representation, but later considered this inadequate and gave an arithmetic formulation. In the late nineteenth century a violent attack on nonconstructive methods was led by Leopold Kronecker in Berlin; his conviction was: "God made the integers, all else is the work of man" [**13**, p. 988]. At the turn of the century, Henri Poincaré in Paris strongly advocated constructive methods; interested in applications, he criticized the classical approach, saying, "True mathematics is that which serves some useful purpose" [**19**].

Although classical mathematics has long been dominant, it was severely challenged during the early part of the century. The challengers, led by L. E. J. Brouwer in Amsterdam, were critical of current practice and called for a new beginning, a careful reconstruction of the basic mathematical framework. The defense, content with classical methods, was led by David Hilbert in Göttingen.

Brouwer (1881–1966) demonstrated that classical mathematics is deficient in numerical meaning. Beginning in 1907, he devoted much of his life to attacking classical methods whose validity he questioned, showing that these methods did not produce mathematical objects which are explicitly constructed and which ultimately reduce to the integers. Brouwer held that in the "constructive process... lies the only possible foundation for mathematics" [**13**, p. 1200]; [**5**].

Hilbert (1862–1943) was a leader in the development of classical mathematics. His solution of "Gordan's Problem" in 1888 had accelerated the growth of modern mathematics. His proof, however, was nonconstructive. Paul Gordan himself, who had for twenty years tried to solve the problem, exclaimed, "Das ist nicht Mathematik. Das ist Theologie!" [**20**, p. 34]. Leading the classical defense against the constructivists, Hilbert angrily complained that "forbidding a mathematician to make use of the principle of excluded middle is like forbidding an astronomer his telescope or a boxer the use of his fists" [**13**, p. 1204].

Hermann Weyl in Zürich strongly supported Brouwer and the constructivists. He charged nonconstructive proofs with lack of significance and value, saying that classical analysis is "built on sand" [**13**, p. 1203].

The Brouwer-Hilbert debate, and the subsequent work of the Brouwerian school, was devoted more to logical, philosophical, and foundational considerations, than to positive constructive developments. A significant advance in the latter direction was made in 1967 by Errett Bishop (1928–1983) in California; his book [**1**] succeeds in developing a large portion of analysis in a realistic, constructive manner.

The central idea in modern constructive mathematics is

BISHOP'S THESIS: *All mathematics should have numerical meaning* [**1**, p. ix].

The significance of this depends upon two essential conditions. First, classical mathematics is deficient in numerical meaning. Second, it is in fact possible to give most mathematics numerical meaning. The first was demonstrated by Brouwer. Although he and his followers also made a great effort to demonstrate the second, and did constructivize certain isolated portions of mathematics, they unfortunately introduced unnecessary idealistic elements into much of their work. Thus Brouwer's main contribution, of crucial significance to the development of mathematics, was his *critique of classical mathematics*. The second step, the systematic *constructive development of mathematics*, was begun in 1967 by Bishop. A small number of workers now continue this development, in constructive algebra [**12**], [**21**], [**24**], [**26**]; constructive analysis [**2**], [**3**], [**4**], [**6**], [**9**], [**10**], [**15**], [**16**], [**17**]; constructive probability theory [**7**], [**8**], and constructive topology [**22**].

Bishop's maxim, "When a man proves a number to exist, he should show how to find it," expresses the constructivist thesis. His book, written to demonstrate the feasibility of "a straightforward realistic approach to mathematics," rebuilds the basics of analysis in a fully constructive manner and provides the framework and methods for continuing further work "to hasten the inevitable day when constructive mathematics will be the accepted norm" [**1**].

Classical vs. constructive mathematics

To avoid a common misunderstanding, it should be stressed that from the constructivist position, classical mathematics does not appear useless, but merely limited. The limitation is crucial, but not fatal. A theorem proved by classical methods is merely incomplete; the degree of incompleteness varies to both extremes. A classical theorem may exhibit no numerical meaning, and there may be little chance of extracting any numerical meaning from it. At the other extreme is the classical theorem that is constructive as it stands, or becomes constructive after some quite minor reformulations. Situated between these extremes, most classical theorems are neither constructively valid nor completely devoid of numerical meaning. Such a theorem must be significantly modified and rephrased to show its true constructive content, and a considerable amount of work must be done to find a constructive proof. Often several new theorems are obtained, revealing different constructive aspects of the classical theorem which were hidden by its classical presentation.

An example will show how a classical theorem can lead to new constructive theorems. Consider the intermediate value theorem which says that *a continuous curve, which is somewhere below the axis and somewhere above, must somewhere cross the axis*. This theorem is not constructively valid; there is no general finite procedure for constructing a crossing point. Furthermore, a counterexample, of a special type due to Brouwer, convinces us that we will never find such a procedure. What can be done? We certainly do not want to discard such a beautiful theorem!

There are two main methods for salvaging constructive theorems from constructively invalid classical theorems. The first method weakens the conclusion, the second strengthens the hypotheses. In each case we must find a constructive proof. Although the resulting theorem does not sound as strong as the original, it is in a deeper sense much stronger—it has numerical meaning.

Applying these methods to the continuous curve theorem, we first weaken the conclusion. We find that we are able to construct, for any small positive number ε, a point of the curve which has a distance less than ε from the axis.

The numerical content of this constructive theorem is clear. What was the content of the original (constructively invalid) classical theorem? The theorem by itself merely makes a statement; it says that there exists a point of the curve which lies on the axis. In what sense, we ask, does such a point exist? Can we actually find the point? Is there a method for constructing it? The mere statement of the classical theorem does nothing to answer these questions. When we examine the proof to see what is actually proved, we find the following: Assuming that no point of the curve lies on the axis, by a certain series of deductions we obtain a contradiction. Thus "existence" in the classical theorem merely means "nonexistence is contradictory." In sharp contrast, our constructive proof of the modified theorem contains the actual construction of a point.

To obtain a second constructive theorem, we strengthen the hypotheses, adding the condition that the curve is defined by a polynomial. This extra hypothesis, while restricting the scope of the theorem, enables us to obtain the original conclusion in full constructive force. We can construct a point of the polynomial curve and prove that it lies exactly on the axis. Various other constructive theorems are also obtained [**1**, p. 59].

For the intermediate value theorem, however, the first constructivization, which finds a point of the curve within ε of the axis, is more important. This is because of its wider applicability, and because, in general, finding solutions to problems "within ε," rather than "exactly," is quite sufficient. This is especially clear when one considers practical applications of mathematics.

What does the classical proof demonstrate? It shows that the existence of such a continuous curve, together with a proof that no point lies on the axis, would lead to a contradiction. This is a useful aspect of the classical proof; we need not waste time trying to construct such a curve. Thus, while the classical theorem may have no constructive affirmative conclusion, it does have practical (although limited) usefulness in directing further constructive effort.

Thus, far from being useless from a constructive viewpoint, classical mathematics serves as an invaluable guide in building the first stages of a fully constructive mathematics. To a large extent

Errett Bishop

this has already been done. Bishop's book contains a complete constructive development of basic analysis: the real numbers, calculus, complex analysis, metric spaces, measure and integration, linear spaces, and more, are constructivized. Others have extended this work and have also begun to constructivize algebra, topology, and probability. Yet there is still much more classical mathematics in need of constructivization.

After the classical results in a given subject have been constructivized as far as possible, then constructive mathematics proceeds to develop the various constructive aspects of the subject which have been uncovered. This enrichment of a mathematical subject is a result of distinguishing between what is constructive and what is not, and is analogous to the enrichment of mathematics which resulted when mathematicians began to distinguish, for example, between infinite series which were convergent, and those which were not.

The principle of excluded middle

The continuous curve problem illustrates the foremost cause of nonconstructivity and lack of numerical meaning in classical mathematics: widespread use of the principle of excluded middle, which says that any meaningful statement is either true or false.

Great care is required with the "either-or" construction, which appears in the excluded middle, and which may be subject to varied interpretations and meanings. Suppose you are going to lunch with a friend and you wish to know whether to take your umbrella. If, using the principle of excluded middle, he or she tells you "*Either* it will rain this noon *or* it will not," you will find this information useless. On the other hand, if *either* you are told "It will rain this noon," *or* you are told "It will not rain this noon," then you will have received valuable constructive information. (This example may give occasion for reflection upon the sort of information, and the degree of certitude, that is found in mathematics, as compared with, for example, meteorology. In fact, mathematical proofs do give predictions. They predict that if certain numbers, with certain relationships between them, are used in performing certain calculations, then the resulting numbers will exhibit certain new relationships.)

Some statements, such as "there exists a prime number between 17,000,000,000 and 17,000,000,017", are in fact constructively either true or false. This is because there is a finite procedure for deciding. For this reason, we might be able to use an indirect proof, a "proof by contradiction." If the assumption that there is no prime number in the specified interval leads to a contradiction, then it would be constructively valid to conclude that there does exist such a prime number. The essential ingredient to such an indirect proof is the prior possession of the finite procedure which either finds the desired prime number or proves there is none. The indirect proof shows that the second alternative is contradictory, and thus *predicts* that the finite procedure, when carried out, will lead to the first alternative, the construction of a prime number in the specified interval. It was for such finite situations only that Aristotle formulated his rules of logic, especially the principle of excluded middle. Nonconstructivities arise when these rules are used indiscriminately in modern mathematics, the science of the infinite.

Now recall our continuous curve. The statement "there exists a point of the curve which lies on the axis" admits of no finite method to determine its truth. There are infinitely many points of the curve. Even for a single point there is no finite method for deciding whether it lies on the axis. In contrast to the above statement about prime numbers, here there is no prior finite procedure for determining one of two alternatives, and thus an indirect proof is constructively invalid.

To the constructivist, use of the principle of excluded middle in infinite situations leads only to pseudoexistence, in the sense that nonexistence is contradictory, rather than existence which stems from a construction. It is the latter type of existence which is appropriate to finite man. While the objects of constructive mathematics are solid objects created by finite constructions, many of the objects of classical mathematics appear as disembodied entities born of questionable logical laws.

Applications of mathematics

A constructive proof, which actually constructs a point with certain properties, has a practical advantage over a classical proof, which merely shows that it is unthinkable that the desired point is nonexistent.

Wandering in the Sahara, would we be content with a nonconstructive proof of the existence of an oasis? Would our parched throats be satisfied with a theorem which asserts that water exists somewhere in the desert, but gives us no clue whatever as to where? Or would we prefer a constructive drink? The classical camel tries to reassure us that there does indeed exist an oasis, but it cannot tell us the direction or the distance. The constructive camel, on the other hand, gives us a direction which, while it may not lead exactly to the oasis, will lead as close as desired. It does give us a point of the compass to follow, and an approximate distance. We might ask it to calculate the direction so as to pass within twenty meters of the oasis. Although it could calculate the direction so as to pass within one millimeter of the exact center of the oasis, this calculation, while still finite, would probably take much longer, wasting precious time. In what sense does the classical camel give the direction? The classical camel asserts that the exact direction exists, but it would take it an infinite amount of time to calculate it, even to find a first approximation. Unless our goatskins happen to contain an infinite amount of water, we might find this classical calculation a bit lengthy.

Applications of mathematics are similar to the Sahara problem. No scientist would be content to learn that a solution to his mathematical problem exists, but that there is no way to calculate it. Thus all experience tends to indicate that any mathematics that is applicable must be constructive. Although there seem to be a few applications of nonconstructive mathematics to theoretical physics, it is likely that it will be the constructive content of these applications which will be useful when the theory reaches the point of experimental verification.

If it is the constructive content of mathematics which is applicable, then since so much mathematics currently being done is nonconstructive, why haven't users of mathematics complained? Two facts help to understand this. First, although most mathematicians make no effort to produce constructive results, nevertheless their results often have a very large (and largely hidden) constructive content. It is this hidden constructive content that is useful and applicable, and a major goal of the constructivist program is to make it explicit. The constructive content of a classical theorem, even of its proof, is often sufficient for applications. On the other hand, much current mathematics is hopelessly nonconstructive; it has no numerical meaning and no constructivizations seem possible. Such mathematics is not being applied and will always remain inapplicable. This brings us to the second fact, the time lag between mathematical work and its applications. This can run to centuries, and thus isolates current research from the test of applicability. The mathematics of previous ages is so useful in present applications, that there is a general belief that all current mathematical work will certainly be usable at some time in the future. This belief may be too optimistic.

Numerical meaning

The suggestion that a theorem may be disputed may at first cause some surprise. Mathematics is often presented as a prime example of indisputable knowledge, against which other less certain forms of knowledge are compared. When even mathematical knowledge comes into question, it may seem that all hope is lost. Nevertheless, two points will clarify the situation. First, there is a limited, but crucially important, part of mathematics about which everyone agrees: the integers. Thus, $5 + 7 = 12$ is indeed a good example of indisputable knowledge. The importance of this seemingly small part of mathematics is that constructive mathematics attempts to *build all mathematics upon these solid integers.*

Secondly, serious misunderstanding is caused by differences in interpretation of the meanings of theorems. Superficially, the dispute sounds quite irresolvable. The classicist has a theorem which states that a certain point exists. The constructivist says it does not exist, and even has a counterexample which convinces him that such a point will never be proved to exist. A closer examination shows that the two are using entirely different meanings of "exists." The classicist has indeed proved that the assumption that such a point does not exist leads to a contradiction, and using the principle of excluded middle, has concluded that such a point does exist. Thus by "existence" the classicist merely means "nonexistence is contradictory." On the other hand, the constructivist, when saying that a point exists, means that a procedure has been given by which the point is explicitly constructed.

Thus the dispute is resolved when we consider theorems only in conjunction with their proofs. After all, the statement of a theorem is nothing more than a summary of what has been demonstrated in the proof, using concise (and often misleading) terminology. When both examine the proof, the classicist and constructivist fully agree about what has been proved. Has the controversy vanished into thin air? No! Rather, we have come to the crux of the issue. The crucial question is, What theorems *should* we prove? The classicist says that the theorem just proved settles the problem, and that the constructivist is wasting time with details. The constructivist says that the classicist has a theorem which is splendid as far as it goes, and which points the way to an interesting and useful constructive theorem, but which in itself is incomplete. In the example of the continuous curve, it is not enough to stop when the nonexistence of the point sought is shown to be contradictory; we should continue until we have constructed a point. The construction reduces to the construction of certain integers; it has numerical meaning. Thus the constructivist's answer to the question, "What theorems should we prove?", is given by Bishop's thesis: "Theorems with numerical meaning!"

It is a natural human tendency, a metaphysical impulse, to believe that every meaningful statement must be either true or false. This is understandable, since we are finite beings, and usually speak only of finite matters. But in exploring the mathematical infinite, we might heed Plutarch: "When talking about infinity we are on treacherous ground and we should just try and keep our footing" [**18**]. We can avoid the quicksand of excluded middle and keep to the constructive trail.

A basic metaphysical problem is whether truth exists independently of man. The classical approach to mathematics presumes that truth does exist in itself, perhaps in some Platonic sphere, and we have only to find it. The constructivist believes that mathematics belongs to man, and that we ourselves create it, except for the integers. These integers, which have been created for us, have already blazed the trail for us to follow in our creation of further mathematical truth.

Constructive real numbers

Both geometry and arithmetic are products of man's thought, based on our concepts of space and integer. By emphasizing arithmetic proofs we are asserting that our concept of integer is more reliable. Even in geometry (and related fields, including analysis) we expect more reliable results if the geometric concepts are reduced to arithmetic concepts. We reduce the concept of a point in the plane to the concept of real number; a point has coordinates which are real numbers. Then real numbers are reduced to rational numbers; a real number is generated by a sequence of approximating rational numbers, e.g., the finite portions of an infinite decimal. Finally, a rational number is a ratio of integers. The classicist also reduces points and numbers to integers in this manner. A close examination, however, shows that the classical reduction uses the principle of excluded middle, which leads to properties of points and numbers which are constructively invalid. A striking example of this is the trichotomy of real numbers. This is the classical theorem that says that *for any real number* x, *either* $x < 0$, *or* $x = 0$, *or* $x > 0$. Constructively, it is not true; there is no known general finite procedure which, for each real number x, leads to a proof of one of these three alternatives. Worse, there is a Brouwerian counterexample which shows that we can never expect to find such a procedure.

To see roughly why trichotomy fails constructively, suppose that the real number $x = 0.a_1a_2a_3a_4\ldots$ is given in decimal form. In any finite length of time we can calculate only finitely many of the digits a_i. At some point we might have calculated a million digits and found them all zero. Still, we cannot in general predict whether all the potentially calculable digits will be zero, or whether a nonzero digit might someday appear. Thus we cannot tell whether $x = 0$ or $x > 0$. This example also indicates why the continuous curve theorem discussed above fails constructively. We have no finite process to decide whether the ordinate of a given point is zero or not; we can only calculate approximations to it. Thus, while we can tell whether a point of the curve lies near the axis, we usually cannot tell whether it lies exactly on the axis. In the Sahara problem, the classical camel might tell us to head west if $x = 0$, but to head east if $x > 0$. Since there are infinitely many decimal digits, we might have to perform an infinitely long calculation before we could take even the first step.

The idea of the trichotomy certainly arises intuitively when we draw a line, mark a zero point on it, and look at the picture. However, this geometric picture of the real numbers, though useful, can be misleading. If we look at a line we may be tempted to think only of points which we deliberately mark off on it, and for which we have a preconceived notion about whether they lie to the left, right, or at the zero point. Thus the trichotomy may seem evident, but this naive view does not take into account all real numbers which have already been constructed, or which may be constructed in the future.

The inadmissibility of trichotomy may seem a mighty blow to constructivism, since it seems to be such a fundamental property of the real numbers. On the contrary, our fondness for the trichotomy arises only from its habitual use; it is not essential for constructive analysis. In its place we use other properties of the real numbers which are constructively valid. For instance, given any small positive number ε, it is constructively true that every real number is either less than ε or greater than zero; we have a dichotomy with a small overlap. Such a dichotomy can be used to construct a point on our continuous curve within a distance ε from the axis.

The relation of constructive mathematics to the whole of mathematics

Most classicists view constructive mathematics as a special, rather minor part of mathematics which, for reasons unclear, avoids the use of certain logical principles and methods of proof. Some followers of Brouwer maintain that constructive mathematics forms a separate branch of mathematics, alongside of and distinct from classical mathematics [**11**, p. 4]. The position of modern constructivists differs from each of these. To them it is classical mathematics which is part of the totality of mathematics; this totality is constructive mathematics. The part which is classical, very large today, but very small in the inevitable future, is that part which uses, as an extra hypothesis, the principle of excluded middle. This extra hypothesis has its limited use, as noted above, but its general use is totally unwarranted.

Often constructive mathematics is considered part of formal logic, philosophy, or the foundations of mathematics. Our point of view, however, is that formal logic and philosophy make only an attempt to lay a "foundation" for mathematics. The practice of mathematics requires no foundation other than that it be based on finite constructions which ultimately reduce to the integers.

In the recent history of constructivism, two main questions arise. After Brouwer's critique of classical mathematics, why was nonconstructive mathematics not immediately rejected? And now with the basic part of mathematics already constructivized, and with methods for further constructive progress at hand, why do only a few use these constructive methods?

The answer to the first question is that although Brouwer's critique of classical mathematics was clear, compelling, widely discussed, and accepted as devastating, still no one had a solution to the problem. Brouwer showed the lack of numerical meaning in classical mathematics, but did not convincingly show how mathematics might be done constructively. In fact, Brouwer himself and his followers were convinced that it was not possible to rebuild most mathematics constructively. They thought that most of the beautiful structures of mathematics would be necessarily lost, and they were willing to suffer this loss for the sake of constructibility [**11**, p. 11]. It was not until later, in 1967, that Bishop showed that there need be no loss, but rather a gain of clarity, precision, applicability, and beauty.

The second question is much more difficult. We can only tentatively suggest some possible reasons for the slow growth of constructivism in mathematics since 1967. Some inertia in the university system and the graduate curriculum may be a factor. Connected with this is the heavy burden on students to master an extensive curriculum which does not always leave sufficient time for exploring new ideas. Another factor may be the chilling effect that the political troubles of this century have had on independent thought. The slow growth of constructive mathematics may also be related to the decay of the idea that the purpose of mathematics is to serve the sciences. Finally, many have been diverted from a careful investigation into the meaning of their own work by the existence of those branches of mathematics which attempt to lay a foundation for the rest of mathematics, validate it, and give it meaning. It is not at all clear that this attempt will succeed. If meaning is to be found in a piece of mathematics, *that* is where it will be found. When you prove a theorem, you must yourself show where the meaning lies; you cannot leave this task to others.

Truth is said to reside in a deep well [**14**]. Reach not for the jug of excluded middle to slake a false thirst; strive to draw what is truly needed from the well.

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