AN EXAMPLE IN CONNECTIVITY

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Bridges [2] has asked whether a located, connected set on the line contains, with any two points \(a\) and \(b\), the closed interval \([a, b]\). In this note a negative answer is given by means of a Brouwerian counterexample [1, pp. 4–5, 25–26]. In this context, a connected set \(S\) is one such that any open, closed, located subset of \(S\) is all of \(S\). Other definitions and required basic results are given in [1]. The counterexample relates the question to the limited principle of omniscience (LPO) [1; p. 9], which we use in the following form: If \(\alpha\) is a real number with \(\alpha \geq 0\), then either \(\alpha > 0\) or \(\alpha = 0\). Since LPO is not true, and an affirmative answer to the question in [2] implies LPO, we conclude that it also is not true. The counterexample depends on the resolution of a bounded colocated set into a union of disjoint open intervals, given in [3] and [4].

Example. "If \(S\) is a located, connected set on the line, and \(a\) and \(b\) are points of \(S\), then \(S\) contains \([a, b]\)" implies LPO.

Proof. Put \(S = (0, 1) \cup \{0, 1\}\). Since \(S\) is dense in \([0, 1]\), which is located, \(S\) itself is located. Let \(F\) be an open, closed, located subset of \(S\). Since \(S\) is totally bounded, \(F\) is totally bounded, hence located in \(\mathbb{R}\). Put \(G = (-\infty, 0] \cup F \cup [1, +\infty)\); then \(G\) is also located. Thus \(U = -G\) has a resolution into disjoint open intervals. Let \(x \in (0, 1)\) and suppose \(\rho(x, F) > 0\); then also \(\rho(x, G) > 0\), so \(x \in U\). Let \(I = (c, d)\) be the component of \(x\) in \(U\). Since \(F\) is located, we may construct a point \(y\) in \(F\). We may assume \(y < d\) (the other case being \(y > c\)); it follows that \(y \leq c\). Since \(F\) is open in \(S\) there exists \(\delta > 0\) such that \(S \cap (y-\delta, y+\delta) \subset F\). (1) If \(c > y\) then \(c > 0\) so \(c\) is in \(S\), hence, since \(F\) is closed in \(S\) and \(I\) is a component of \(U\), we have \(c\) in \(F\). Since \(F\) is open, \(c\) has a neighbourhood contained in \(F\), which must meet \(U\), a contradiction. (2) If \(c < y + \delta\), then there exists \(z\) such that \(c < z < (y + \delta) \land d\). Then \(y < z < y + \delta\) so \(z \in F\), and also \(z \in U\), a contradiction. It follows that \(\rho(x, F) = 0\), so \(x \in F\). Thus \((0, 1) \subset F\), and \(F = S\). Hence \(S\) is connected. By hypothesis, \(S\) contains \([0, 1]\). Now let \(\alpha > 0\). It suffices to consider the case \(\alpha < 1\); then \(\alpha \in S\), so either \(\alpha > 0\) or \(\alpha = 0\).

Corollary. The set \((0, 1) \cup \{0, 1\}\) is connected.

References


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Received 9 March, 1978