AN EXAMPLE IN CONNECTIVITY

MARK MANDELKERN

Bridges [**2**] has asked whether a located, connected set on the line contains, with any two points a and b, the closed interval $[a, b]$. In this note a negative answer is given by means of a Brouwerian counterexample [**1**, pp. 4–5, 25–26]. In this context, a *connected* set S is one such that any open, closed, located subset of S is all of S. Other definitions and required basic results are given in [**1**]. The counterexample relates the question to the limited principle of omniscience (LPO) [**1**; p. 9], which we use in the following form: If α is a real number with $\alpha \geqslant 0$, then either $\alpha > 0$ or $\alpha = 0$. Since LPO is *not* true, and an affirmative answer to the question in [**2**] implies LPO, we conclude that it also is *not* true. The counterexample depends on the resolution of a bounded colocated set into a union of disjoint open intervals, given in [**3**] and [**4**].

EXAMPLE. "*If S is a located, connected set on the line, and a and b are points of S, then S contains $[a, b]$*" *implies LPO.*

Proof. Put $S \equiv (0, 1) \cup \{0, 1\}$. Since S is dense in $[0, 1]$, which is located, S itself is located. Let F be an open, closed, located subset of S. Since S is totally bounded, F is totally bounded, hence located in R. Put $G \equiv (-\infty, 0] \cup F \cup [1, +\infty)$; then G is also located. Thus $U = -G$ has a resolution into disjoint open intervals. Let $x \in (0, 1)$ and suppose $\rho(x, F) > 0$; then also $\rho(x, G) > 0$, so $x \in U$. Let $I = (c, d)$ be the component of x in U. Since F is located, we may construct a point y in F. We may assume $y < d$ (the other case being $y > c$); it follows that $y \leqslant c$. Since F is open in S there exists $\delta > 0$ such that $S \cap (y-\delta, y+\delta) \subset F$. (1) If $c > y$ then $c > 0$ so c is in S, hence, since F is closed in S and I is a component of U, we have c in F. Since F is open, c has a neighbourhood contained in F, which must meet U, a contradiction. (2) If $c < y+\delta$, then there exists z such that $c < z < (y+\delta) \wedge d$. Then $y < z < y+\delta$ so $z \in F$, and also $z \in U$, a contradiction. It follows that $\rho(x, F) = 0$, so $x \in F$. Thus $(0, 1) \subset F$, and $F = S$. Hence S is connected. By hypothesis, S contains $[0, 1]$. Now let $\alpha \geqslant 0$. It suffices to consider the case $\alpha < 1$; then $\alpha \in S$, so either $\alpha > 0$ or $\alpha = 0$.

COROLLARY. *The set* $(0, 1) \cup \{0, 1\}$ *is connected.*

References

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3. M. Mandelkern, "Connectivity of an interval," *Proc. Amer. Math. Soc.*, 54 (1976), 170–192.
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Mathematics Department,
Birkbeck College,
London W.C.1.,
England.

Mathematics Department,
New Mexico State University,
Las Cruces,
New Mexico 88003.

Received 9 March, 1978