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Finitary Sequence Spaces

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Abstract

This paper studies the metric structure of the space H^r of absolutely summable sequences of real numbers with at most r nonzero terms. H^r is complete, and is located and nowhere dense in the space of all absolutely summable sequences. Totally bounded and compact subspaces of H^r are characterized, and large classes of located, totally bounded, compact, and locally compact subspaces are constructed. The methods used are constructive in the strict sense.
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1. Introduction

1.1. Many developments in mathematics may be described as continual refinements by means of finer and finer distinctions. Thus, the irrational and transcendental, the continuous and differentiable, and so forth. The constructivist tradition may easily be viewed in this manner. Eschewing dependence on classical tenets, such as the principle of excluded middle, we find that concepts often split into several quite distinct fragments. Here we explore one aspect of the question – What is Finite? – from a constructive viewpoint.

The usual notion of a finite set entails correspondence with an initial segment of the positive integers. What then is an infinite series with only finitely many non-zero terms? Are we given a finite set of r indices, the corresponding non-zero terms, and a guarantee that all other terms are zero? Or may some of the specified terms also be zero? Or is there given merely an upper bound r for the number of nonzero terms? In which case, deprived of classical omniscience, we cannot determine where the nonzero terms might be situated! Or do we have a finite upper bound for the indices of the non-zero terms?

We find that by adopting the appropriate definition we may construct a family $\{H^r\}$ of complete subspaces of the space H of all absolutely summable sequences. We shall explore the metric structure of the spaces H^r, and of their totally bounded, compact, and locally compact subspaces.

How did the problem of finitary sequence spaces arise? A counterexample space often used in topology is known as the "Hedgehog". This metric space J is the union of countably many copies of the closed unit interval $I \equiv [0,1]$, with all the points 0 identified. In the course of constructivizing this space, a problem arises; although J is classically complete, the proof is not constructive. To obtain a constructively complete space, a modified construction is required.

The original definition of the Hedgehog may be viewed as follows: To construct a point of J, choose a number α in the interval, and an index k to indicate which copy of the interval is intended. It is possible to define a Cauchy sequence of such points whose (classical) limit cannot be found constructively. To describe the counterexample very roughly, the points change index so rapidly, and approach zero so closely, that one cannot see whether they tend to zero, or stop short of zero in one of the interval copies, whose index cannot be observed. (The precise Brouwerian counterexample is given below in Example 2.2.)

To resolve this problem, we notice that the specification of the index k is necessary only when the number α is positive. Thus a classically equivalent construction of the space J is the set of all sequences in I with at most one nonzero term. The previous (multiple) point 0 becomes the constantly 0 sequence. The point α in the k^{th} interval becomes the sequence $x \equiv (x_n)$ with $x_k = \alpha$, and $x_n = 0$ for $n \neq k$. Classically, no new points are introduced. Constructively, however, new points are provided: the limits of Cauchy sequences which previously had no constructive limits. Such a point is a sequence in I with at most one nonzero term, but for which we can *neither* specify a nonzero term, *nor* prove there is none. Here is a simple example: Conduct a sequential search for a counterexample to the Goldbach Conjecture. If the conjecture is verified up to the $(k-1)^{th}$ step, define x^k to be the point of J with value $1/k$ in the k^{th} interval. However, if the first counterexample is found at the j^{th} step, define $x^k \equiv x^j$ for all $k > j$. Then $\{x^k\}$ is a Cauchy sequence which has no limit in the original space J. (Brouwerian counterexamples of this sort are discussed more precisely in [1], [2], and [10].) Now define the sequence $x \equiv (x_n)$ by $x_n \equiv 1/n$ if the first counterexample is found at the n^{th} step, and $x_n \equiv 0$ otherwise. Then x is a point of the space J defined by sequences, and $x^k \to x$.

The classical Hedgehog J is described as the set of pairs (α, k), where $\alpha \in I$ and $k \in \mathbb{Z}^+$, with all pairs of the form $(0,k)$ identified, and with metric ϱ defined by

$$\varrho((\alpha,k),(\beta,j)) \equiv \begin{cases} |\alpha-\beta| & \text{if} \quad k=j, \\ \alpha+\beta & \text{if} \quad k \neq j. \end{cases}$$

With the sequence definition of J this may be written simply as

$$\varrho(x,y) \equiv \sum_{n=1}^{\infty} |x_n - y_n| \, .$$

Now, however, since we do not require knowledge of the location of the possibly nonzero term (as was necessary to obtain a complete space), we have introduced a problem in constructively defining the metric. We must require summability of the sequence which determines a point of J. As absurdly trivial as it may seem to sum a sequence which is all zeros except for at most one term, the constructive necessity of this condition becomes clear when one substitutes 1 for $1/n$ in the example just given above. If it were possible to estimate the sum of the resulting sequence, then Goldbach's Problem would be solved.

We generalize the Hedgehog by considering, for any positive integer r, the space H^r of absolutely summable sequences of real numbers with at most r nonzero terms. The original Hedgehog J [4, Ex. 4.1.5] is realized as a located subspace of H^1 in Lemma 3.3 below.

1.2. Summary of results. The results obtained here are constructive in the sense of ERRETT BISHOP's *Foundations of Constructive Analysis* [1]. One exposition of the constructivist thesis may be found in [8].

This paper studies the metric structure of H^r. The space H^r is complete, and is located and nowhere dense in the space H of all absolutely summable sequences. Totally bounded and compact subspaces of H^r are characterized. The essential characteristic of a totally bounded subspace of H^r is a common sequence of summability parameters for the sequences in the subspace. Large classes of located, totally bounded, and compact subspaces are constructed. Although the spaces H^r are classically σ-compact, none is constructively σ-compact. The space H^r is not locally compact, but does contain a dense, locally compact subspace L^r, consisting of sequences with exactly r nonzero terms.

A number of Brouwerian counterexamples have been included where appropriate, in order to determine as closely as possible the constructive limits of the theory. This method is described in [1], [2], and [10]. Most of the examples are given in a form which shows a certain statement nonconstructive because it is *equivalent* to one of a number of nonconstructive omniscience principles. Since these omniscience principles are classically valid, these counterexamples are at the same time classical proofs of the propositions involved.

Projected studies will consider the characterization and construction of real-valued continuous functions on H^r, and of continuous mappings between the various spaces H^r, and between H^r and the Euclidean spaces.

1.3. Preliminaries. For real numbers α we distinguish between the concepts "α is not zero" and "α is nonzero". The first means that it is contradictory that α be zero. The second, which is much stronger, and more often used, is written $\alpha \neq 0$, and means $|\alpha| > 0$. The condition $\beta > 0$ means that for some positive integer k, an explicit rational approximation q has been found, to within $1/k$ of β, with $q > 1/k$.

In lieu of the trichotomy of real numbers, which is not constructively valid, we will make frequent use of the *Constructive Dichotomy Lemma*: If $\alpha < \beta$, then for any x, either $x < \beta$ or $x > \alpha$. See [1, p. 24], [2, p. 26] or [10, p. 16].

A *finitely enumerable set of at most r elements*, where $r \geq 1$, is a set of the form $A \equiv \{a_1, a_2, \ldots, a_r\}$. If, in addition, a_i is not a_j whenever $i \neq j$, then A is called a *finite set of r elements*. The void set is not considered finite. Other basic constructive notions may be found in [1] or [2].

2. Basic Structure of H^r

2.1. Definitions. Let H denote the set of all absolutely summable sequences of real numbers. Then H is a metric space, with distance ϱ defined by

$$\varrho(x,y) \equiv \sum_{n=1}^{\infty} |x_n - y_n|$$

for any points $x \equiv (x_n)_{n=1}^{\infty}$ and $y \equiv (y_n)_{n=1}^{\infty}$. The (linear) space H is usually referred to as ℓ^1, although the subspaces of H considered here will not be linear. The usual proof that H is complete, a simple exercise, is constructively valid.

We are interested in points of H with at most finitely many nonzero terms, and must make this notion precise. For any integer $r \geq 1$, it would seem natural to consider the subspace G^r consisting of all points $x \equiv (x_n)$ in H such that there exists a finite set P of r positive integers with $x_n = 0$ for all $n \notin P$. However, this space G^r is not constructively complete (see Example 2.2 below). A weakening of the condition does yield a constructively larger space which is complete. Let H^r denote the subspace consisting of all points $x \equiv (x_n)$ in H such that whenever P is a finite set of r positive integers with $x_n \neq 0$ for all $n \in P$, then $x_n = 0$ for all $n \notin P$. Equivalently, a point $x \equiv (x_n)$ of H lies in H^r if and only if, whenever P is a finite set of positive integers with $x_n \neq 0$ for all $n \in P$, then card $P \leq r$. Any sequence of real numbers satisfying this condition will be called *finitary of order r*.

We will show that H^r is complete, and that G^r is a dense subspace. Thus the two similar, classically equivalent, conditions used to define these spaces are constructively distinct. For a point of H^r the number of nonzero terms is limited by r, while for a point of G^r the r indices of the possibly nonzero terms must be specified. Finally, let F^r denote the subspace consisting of all points $x \equiv (x_n)$ of G^r such that x_n is rational for all n. It will also be convenient to denote by H^0 the subspace $\{0\}$, where 0 denotes the constantly zero sequence.

We first show that the spaces G^r are not constructively complete, demonstrating the need to consider H^r. For a discussion of the nonconstructive omniscience principles such as *The Limited Principle of Omniscience* (LPO), see [10] or [7, Sec. 2].

2.2. Example. *For any $r \geq 1$, the statement "G^r is complete" is nonconstructive; it is equivalent to* LPO.

Proof. First assume that G^r is complete. To derive LPO, let $\{a_k\}$ be a decision sequence (a nondecreasing sequence of zeros and ones); we must show either that all $a_k = 0$ or that some $a_k = 1$. We may assume that $a_r = 0$. Define a sequence $\{x^k\}_{k=r}^{\infty}$ in G^r as follows. If $a_k = 0$, let x^k be the point of G^r with $x_n^k \equiv 1$ for all $n < r$, with $x_k^k \equiv 1/k$, and with $x_n^k \equiv 0$ otherwise. If $a_k = 1$, define $x^k \equiv x^{k-1}$. It is easy to verify that $\{x^k\}$ is a Cauchy sequence in G^r. By hypothesis, $x^k \longrightarrow y$ for some point y of G^r, so there is a set P of r positive integers such that $y_n = 0$ for all $n \notin P$. It follows that P is of the form $P = \{1, 2, \ldots, r-1, q\}$, with $q \geq r$. Either $y_q < 1/q$ or $y_q > 0$. In the first case, consider any k and suppose that $a_k = 1$. Then there is an integer $j \geq r$ such that $a_j < a_{j+1}$, and $x^k = x^j$ for all $k > j$. Thus $y = x^j$, and $y_j = x_j^j = 1/j$. It follows that $j = q$, a contradiction; thus all $a_k = 0$. In the second case, where $y_q > 0$, there exists $k > q$ such that $x_q^k > 0$, and thus $a_k = 1$.

Conversely, let $\{x^k\}$ be a Cauchy sequence in G^r. Since H is complete, $x^k \to x$ for some $x \in H$. Using LPO, we may determine whether $x_n = 0$ or $x_n \neq 0$ for each n; define $P \equiv \{n : x_n \neq 0\}$. Since $x_n^k \to x_n$ for each n, it follows that P is finite, with card $P \leq r$; thus $x \in G^r$. □

We now show that, for finitary sequences, convergence to zero ensures summability.

2.3. Lemma. *Let $x \equiv (x_n)$ be a finitary sequence of real numbers, of order r. Then the following are equivalent:*

(i) $x_n \to 0$.

(ii) $\sum x_n$ *is absolutely convergent (and thus $x \in H^r$).*

Proof. Let $\varepsilon > 0$ and choose N so that $|x_n| < \varepsilon/2r$ for all $n \geq N$. Let $M > N$ and partition $\{N, N+1, \ldots, M\} = T \cup P$ so that $|x_n| < \varepsilon/2M$ for all $n \in T$ and $|x_n| > 0$ for all $n \in P$. It follows that card $P \leq r$, and thus $\sum_{n=N}^{M} |x_n| = \sum_{n\in T} |x_n| + \sum_{n \in P} |x_n| < M(\varepsilon/2M) + r(\varepsilon/2r) = \varepsilon$. □

2.4. Examples. (1) Let $\{a_n\}$ be a sequence of zeros and ones, with at most a single one, for which we do *not* know whether some $a_n = 1$. While $\{a_n\}$ is finitary, we do *not* know that $a_n \to 0$, *nor* that $\sum a_n$ converges. On the other hand, $a_n/n \to 0$, and thus $\sum a_n/n$ converges.

(2) Let $\{a_n\}$ be a sequence of zeros and ones, such that "all $a_n = 0$" is contradictory, but, constructively, we do *not* know when $a_n = 1$. Define $x_n \equiv 1/n$, unless $a_k = 1$ for some $k \leq n$, in which case define $x_n \equiv 0$. Then $x_n \to 0$. Classically, we do know when $a_n = 1$, so $\{x_n\}$ is classically finitary, and $\sum x_n$ converges classically. However, $\{x_n\}$ is *not* constructively finitary, and $\sum x_n$ is *not* constructively convergent.

(3) Let $\{a_n\}$ be a sequence of zeros and ones, with at most a single one, for which we do *not* know whether some $a_n = 1$, and let $T \equiv \{a_n : n \in \mathbb{Z}^+\}$ be the set of its terms. Then T has at least one, and at most two, elements, but T is not finitely enumerable according to constructive usage.

(4) There is no constructive procedure for deciding, given a real number α, whether $\alpha = 0$ or $\alpha \neq 0$. Let $r > 1$ and let $x \equiv (x_n)$ be a point of H^r with $x_1 = 1$. Then the set $A \equiv \{n \in \mathbb{Z}^+ : x_n \neq 0\}$ is nonvoid, but A need not be finite. Even when x is a point of G^r, so that A is a nonvoid subset of a finite set of r positive integers, A need not be finite.

(5) A subset B of a set A is said to be *detachable* if for each point a in A, either $a \in B$ or $a \notin B$. The statement "Every nonvoid detachable subset of the positive integers is either finite or infinite" is nonconstructive; it is equivalent to LPO. This is proved, not quite explicitly, in [9]. The set A of Example (4) is not, in general, detachable. When the terms of x are rational, then A is nonvoid and detachable, and not infinite, but still A is not, in general, finite.

2.5. Theorem. *Each space H^r is complete.*

Proof. Let $\{x^k\}_{k=1}^{\infty}$ be a sequence of points in H^r with $x^k \to y$, where $y \equiv (y_n)$ is a point of H. Let P be a finite set of positive integers such that $y_n \neq 0$ for all $n \in P$, define $\delta \equiv \min\{|y_n| : n \in P\}$, and choose a positive integer N so that $\varrho(x^N, y) < \delta$. Then $x_n^N \neq 0$ for all $n \in P$, and it follows that card $P \leq r$. Thus H^r is closed in H. □

2.6. Lemma. *For any r, the subspace F^r is dense in H^r.*

Proof. Let $x \equiv (x_n)$ be a point of H^r, and let $\varepsilon > 0$. Choose a positive integer N so that $\sum_{n>N} |x_n| < \varepsilon/3$. Partition $\{1, 2, \ldots, N\} = P \cup T$ so that $|x_n| > 0$ for all $n \in P$ and $|x_n| < \varepsilon/3N$ for all $n \in T$. For all $n \in P$, choose a rational number y_n so that $|x_n - y_n| < \varepsilon/3r$, and define $y_n \equiv 0$ otherwise. Since card $P \leq r$, the point $y \equiv (y_n)$ lies in F^r. Clearly $\varrho(x, y) < \varepsilon$. □

There is a third condition, distinct from the two so far used, which may be imposed on points of H:

2.7. Example. *Let $r \geq 1$ and let D^r be the subspace consisting of all points $x \equiv (x_n)$ in H^r such that there exists a positive integer p such that $x_n = 0$ for all $n > p$. Then $G^r \subset D^r \subset H^r$, but the statements "$D^r = H^r$" and "$G^r = D^r$" are nonconstructive; the first is equivalent to* LPO *and the second to* LLPO.

Proof. The case $r = 1$ will suffice to indicate the proof for any r. Assume that $D^1 = H^1$ and let $\{a_n\}$ be a decision sequence. Define $x_n \equiv 1/n$ if $a_n < a_{n+1}$, and $x_n \equiv 0$ otherwise. Then $x \equiv (x_n)$ is a point of H^1, so by hypothesis there is a positive integer p such that $x_n = 0$ for all $n \geq p$. If $a_p = 0$, then $a_n = 0$ for all n; thus LPO follows. The converse follows from Example 2.2 and Lemma 2.6.

Now assume that $G^1 = D^1$. We will establish LLPO (*The Lesser Limited Principle of Omniscience*) in the following form: *For any real number a, either $a \geq 0$, or $a \leq 0$.* Define $x_1 \equiv a \vee 0$, $x_2 \equiv (-a) \vee 0$, and $x_n \equiv 0$ for all $n \geq 3$. This defines a point x of D^1; by hypothesis there is a positive integer p such that $x_n = 0$ for all $n \neq p$. If $p = 1$ then $a \geq 0$, while if $p \neq 1$ then $a \leq 0$; thus LLPO follows. Conversely, given $x \in D^1$ we may use LLPO to choose a positive integer q such that $|x_q| \geq |x_n|$ for all n. Then $x_n = 0$ for all $n \neq q$, and it follows that $x \in G^1$. □

3. Located and totally bounded subspaces

A metric space (X, ϱ) is *totally bounded* if it contains a finite ε approximation for any $\varepsilon > 0$; equivalently, if it contains finitely enumerable approximations. A metric space is *compact* if it is totally bounded and complete. A subset Y of a metric space X is *located* if the distance $\varrho(x, Y)$ to the subset may be measured from any point x in X. Compact subspaces are always located. The *metric complement* $X - Y$ of a located subset Y is the set of points in X situated at a positive distance from Y. These and other constructive properties of metric spaces are developed in [1] or [2]. We will also need the following elementary properties of located sets [5]: The closure of a located subset of a metric space is also located. A dense subset of a located subset is also located. If Y is a subset of a metric space X, and $\varrho(x, Y)$ exists for all points x in some dense subset of X, then Y is located in X. A method for constructing all located sets on the real line is given in [6].

While classically every subset of a metric space is located, in a constructive study it is a central concern to identify sufficiently many located sets. An example is the construction of compact subspaces of H^r in Theorem 4.1 below, using Lemma 3.3, which constructs a large class of located sets. We require first a routine calculation.

3.1. Definition. Let A be a finitely enumerable set of at most N real numbers, and let $\varepsilon > 0$. An *ε approximate ordering of A* is an enumeration $A = \{a_1, a_2, \ldots, a_N\}$ in which $a_i < a_j + \varepsilon$ whenever $1 \le i < j \le N$.

3.2. Proposition. *Every finitely enumerable set of real numbers has ε approximate orderings for all $\varepsilon > 0$.*

Proof. If $N = 2$ and $A = \{b, c\}$, either $b < c$ or $b > c - \varepsilon$; define $a_1 = b$ or $a_1 = c$ accordingly. If $N > 2$, write $A = B \cup \{b\}$, and by induction construct an $\varepsilon/2$ approximate ordering $B = \{a_1, a_2, \ldots, a_{N-1}\}$. For each i with $1 \le i \le N-1$, determine either $b > a_i - \varepsilon/2$ or $b < a_i + \varepsilon/2$, and define $\sigma_i = 0$ or $\sigma_i = 1$ accordingly. If all $\sigma_i = 0$, define $a_N = b$; if all $\sigma_i = 1$, define $a_0 = b$. If $\sigma_i \ne \sigma_{i+1}$ for some i, then $\{a_1, a_2, \ldots, a_i, b, a_{i+1}, \ldots, a_{N-1}\}$ is an ε approximate ordering of A. □

3.3. Main Lemma. *Let $\{X_n\}$ be a sequence of located subsets of $\mathbb{R}$, with $0 \in X_n$ for each n. Then the subspace $X \equiv \{x \in H^r : x_n \in X_n$ for all $n\}$ of H^r is located in H.*

Proof. Let $y \in H$, and for each $k \ge r$, partition $\{1, \ldots, k\} = L_k \cup S_k$ so that $\operatorname{card} L_k = r$ and $|y_n| - \varrho(y_n, X_n) > |y_m| - \varrho(y_m, X_m) - 1/k$ whenever $n \in L_k$ and $m \in S_k$. Define

$$d_k \equiv \sum_{n \in L_k} \varrho(y_n, X_n) + \sum_{n \notin L_k} |y_n|;$$

thus $d_k = \sum_{n=1}^{\infty} \varrho(y_n, X_n) + \sum_{n \notin L_k} [|y_n| - \varrho(y_n, X_n)]$. Let $\varepsilon > 0$ and choose N so that $1/N < \varepsilon$, and so that $\delta_n \equiv |y_n| - \varrho(y_n, X_n) < \varepsilon$ for all $n > N$. If $j > k \ge N$, then L_j is obtained from L_k by replacing at most r indices n, corresponding to each of which the change in δ_n is less than ε; thus $|d_k - d_j| < r\varepsilon$. Hence $\{d_k\}$ is a Cauchy sequence; define $d \equiv \lim_k d_k$.

Now let $x \in X$, and let $\varepsilon > 0$. It is not difficult to construct a point w in $X \cap G^r$ with $\varrho(x, w) < \varepsilon$ (cf. Lemma 2.6). Choose a set Q of r positive integers such that $w_n = 0$ for all $n \notin Q$. Choose a positive integer k so that $k \ge n$ for all $n \in Q$, so that $1/k < \varepsilon$, and so that $d_k > d - \varepsilon$. Then

$$\begin{aligned}
\varrho(y, w) &= \textstyle\sum_{n=1}^{\infty} |y_n - w_n| \\
&= \textstyle\sum_{n \in Q} |y_n - w_n| + \sum_{n \notin Q} |y_n| \\
&\ge \textstyle\sum_{n \in Q} \varrho(y_n, X_n) + \sum_{n \notin Q} |y_n| \\
&= \textstyle\sum_{n=1}^{\infty} \varrho(y_n, X_n) + \sum_{n \notin Q} [|y_n| - \varrho(y_n, X_n)] \\
&\ge \textstyle\sum_{n=1}^{\infty} \varrho(y_n, X_n) + \sum_{n \notin L_k} [|y_n| - \varrho(y_n, X_n)] - r/k \\
&= d_k - r/k > d - (r+1)\varepsilon.
\end{aligned}$$

Thus $d < \varrho(y, x) + (r+2)\varepsilon$.

Finally, for any k, choose $z_n \in X_n$ so $|y_n - z_n| < \varrho(y_n, X_n) + 1/k$ for all $n \in L_k$ and define $z_n \equiv 0$ otherwise. Then $z \in X$ and

$$\varrho(y, z) = \sum_{n=1}^{\infty} |y_n - z_n| < \sum_{n \in L_k} \varrho(y_n, X_n) + r/k + \sum_{n \notin L_k} |y_n| = d_k + r/k.$$

This shows that $\varrho(y, X) = d$. □

Remark. Taking $r = 1$ and $X_n = [0,1]$ for all n, the original Hedgehog J is realized as the located subspace $X \cap G^1$.

3.4. Theorem. *Each space H^r is located in H.* □

3.5. Example. *For any $r \geq 1$, the statement "For any point y of H, there exists a point z of H^r such that $\varrho(y, H^r) = \varrho(y,z)$" is nonconstructive; it is equivalent to* LLPO.

Proof. It will suffice to give the proof for $r = 1$. Let $a \in \mathbb{R}$, and define $y \equiv (1+a, 1-a, 0, \ldots)$. By hypothesis, choose $z \in H^1$ such that $\varrho(y, H^1) = \varrho(y,z)$. It will now be convenient to derive certain consequences of implications involving the conditions $a > 0$ and $a < 0$, although of course we have no method for obtaining such determinations. If $a > 0$, then it is clear that $z = (1+a, 0, \ldots)$; so $z_1 > 1$. Thus $z_1 < 1$ implies $a \leq 0$. Similarly, if $a < 0$, then $z = (0, 1-a, 0, \ldots)$, and $z_1 = 0$. Thus $z_1 > 0$ implies $a \geq 0$. Since we can determine either that $z_1 < 1$ or that $z_1 > 0$, it follows that either $a \leq 0$ or $a \geq 0$.

Conversely, assume LLPO and let $y \in H$. Construct a decision sequence $\{\sigma_k\}$ such that $\varrho(y, H^1) < 1/k$ when $\sigma_k = 0$ and $\varrho(y, H^1) > 0$ when $\sigma_k = 1$. Define a sequence $\{z^k\}$ in H^1 as follows. When $\sigma_k = 0$, choose a point z^k in H^1 so that $\varrho(y, z^k) < 1/k$.

When $\sigma_k = 1$, we will first define z^k in the case where k is the least such integer, and then define $z^j \equiv z^k$ for all $j > k$. Choose a positive integer m such that $|y_m| > 0$, and choose N so that $|y_n| < |y_m|$ for all $n > N$. Using LLPO, choose $p \leq N$ so that $|y_p| \geq |y_n|$ for all n, and define z^k to be the point of H^1 with $z^k_p \equiv y_p$. Let $x \in G^1$ and choose q so that $x_n = 0$ for all $n \neq q$. Then $\varrho(y,x) \geq \sum_{n \neq q} |y_n| \geq \sum_{n \neq p} |y_n| = \varrho(y, z^k)$. It follows (in this case) that $\varrho(y, z^k) = \varrho(y, H^1)$. This defines the sequence $\{z^k\}$.

Now let $k < j$. If $\sigma_j = 0$, then $\varrho(z^k, z^j) < 2/k$. If $\sigma_k = 1$, then $z^k = z^j$. Finally, if $\sigma_k = 0$ and $\sigma_j = 1$, then $\varrho(y, z^k) < 1/k$ and $\varrho(y, z^j) = \varrho(y, H^1) < 1/k$; thus $\varrho(z^k, z^j) < 2/k$. Hence $\{z^k\}$ is a Cauchy sequence, and $z^k \to z$ for a point z of H^1. Since $0 \leq \varrho(y, z^k) - \varrho(y, H^1) < 1/k$ for all k, it follows that $\varrho(y, H^1) = \varrho(y,z)$. □

3.6. Remark. Using the same method, and the representation of located sets on the line by notches [6], one may also show that the statement "For any complete located set $X \subset \mathbb{R}$, and any point y in $\mathbb{R}$, there exists a point z in X such that $\varrho(y, X) = \varrho(y,z)$" is equivalent to LLPO.

We wish to construct compact subspaces of H^r, and will first characterize the totally bounded subspaces of H^r and H. In any metric space, a totally bounded subspace is bounded and located. For subspaces of the Euclidean spaces, these conditions are also sufficient, although they are usually not sufficient in other spaces.

3.7. Definition. A family X of sequences of real numbers is said to be *equisummable* if for any $\varepsilon > 0$ there exists a positive integer N such that $\sum_{n=N}^{\infty} |x_n| < \varepsilon$ for any sequence $x \equiv (x_n)$ in X.

The proof of the following lemma is virtually the same as that of Lemma 2.3. (A counterexample to the corresponding statement in H is given by the family of sequences of the form $(1/n^p)$, for $p > 1$.)

3.8. Lemma. *A subspace X of H^r is equisummable if and only if for any $\varepsilon > 0$ there exists a positive integer N such that $|x_n| < \varepsilon$ for all points $x \equiv (x_n)$ in X, and all $n \geq N$.* □

3.9. Theorem. *A subspace X of H^r is totally bounded if and only if it is bounded, located, and equisummable.*

Proof. First let X be totally bounded, and let $\varepsilon > 0$. If A is a finite $\varepsilon/2$ approximation to X, and $\sum_{n=N}^{\infty} |a_n| < \varepsilon/2$ for each $a \in A$, then $\sum_{n=N}^{\infty} |x_n| < \varepsilon$ for all $x \in X$.

Conversely, let X satisfy the three conditions, and let $\varepsilon > 0$; we may assume that $\varepsilon < 1$. First choose a positive integer N so that $\sum_{n=N}^{\infty} |x_n| < \varepsilon$ for all $x \in X$, and choose a real number λ so that $\varrho(x,0) < \lambda - 1$ for all $x \in X$. Now choose a finite ε/N approximation B to the interval $(-\lambda, \lambda)$, with $0 \in B$. Define

$$Y \equiv \{y \in H^r : y_n \in B \text{ when } n < N, \text{ and } y_n = 0 \text{ when } n \geq N\}.$$

Then Y is a finite subset of H^r. Since X is located, we may partition $Y = W \cup T$ so that $\varrho(y, X) < 5\varepsilon$ for all $y \in W$ and $\varrho(y, X) > 4\varepsilon$ for all $y \in T$. For each $y \in W$ choose $y' \in X$ such that $\varrho(y, y') < 5\varepsilon$ and define Y' to be the set of all points y' so obtained. Then Y' is a finitely enumerable subset of X.

Now let $x \in X$. Choose $z \in G^r$ so that $\varrho(x, z) < \varepsilon$, and choose a finite set P of r positive integers such that $z_n = 0$ for all $n \notin P$. It follows that $\sum_{n=N}^{\infty} |z_n| < 2\varepsilon$ and $|z_n| < \lambda$ for all n. For each n in $Q \equiv \{n \in P : n < N\}$, choose a number $y_n \in B$ so that $|z_n - y_n| < \varepsilon/N$, and define $y_n \equiv 0$ for all $n \notin Q$. Then the point $y \equiv (y_n)$ lies in Y and $\varrho(y, X) \leq \varrho(x, y) \leq \varrho(x, z) + \varrho(z, y) \leq \varepsilon + \sum_{n<N} |z_n - y_n| + \sum_{n \geq N} |z_n - y_n| < \varepsilon + N(\varepsilon/N) + 2\varepsilon = 4\varepsilon$. It follows that $y \in W$ and $\varrho(x, y') < 9\varepsilon$. Hence Y' is a finitely enumerable 9ε approximation to X. This shows that X is totally bounded. □

3.10. Corollary. *A subspace of H^r is compact if and only if it is closed, bounded, located, and equisummable.* □

Minor changes in the proof of Theorem 3.9 will yield a proof of the following

3.11. Theorem. *A subspace X of H is totally bounded if and only if it is bounded, located, and equisummable.* □

The following proposition shows that, in applying Theorem 3.9 and Corollary 3.10, the boundedness condition may be verified termwise.

3.12. Proposition. *A nonvoid subspace X of H^r is bounded if and only if there is a real number λ such that $|x_n| \leq \lambda$ for all points $x \equiv (x_n)$ in X, and all n.*

Proof. Let the given condition hold, let $x \in X$, and let $\varepsilon > 0$. For any N, partition $\{1, 2, \ldots, N\} = W \cup T$ so that $|x_n| > 0$ for all $n \in W$ and $|x_n| < \varepsilon/N$ for all $n \in T$. It follows that card $W \leq r$, and thus $\sum_{n=1}^{N} |x_n| = \sum_{n \in W} |x_n| + \sum_{n \in T} |x_n| < r\lambda + N(\varepsilon/N) = r\lambda + \varepsilon$. Thus $\varrho(x, 0) \leq r\lambda$ for all $x \in X$. □

4. Compact subspaces

We now consider methods for constructing a compact subspace X of H^r. We begin with a sequence $\{X_n\}$ of compact subsets of the real line, and use the set X_n to restrict the n^{th} term x_n of a point $x \equiv (x_n)$ of X. This may be done in two ways, requiring either that $x_n \in X_n$ for all n, or only that $x_n \in X_n$ whenever $x_n \neq 0$. A third condition, "for each n, either $x_n = 0$ or $x_n \in X_n$", will not produce a constructively complete subspace.

The first, and simplest, method relies on the previous results on located and totally bounded subspaces. (A counterexample, showing that the corresponding construction in H will not produce a compact subspace, may be obtained using sequences of the form $(1/n^p)$, for $p > 1$.)

4.1. Theorem. *Let $\{X_n\}$ be a sequence of totally bounded subsets of $\mathbb{R}$, with $0 \in X_n$ for each n, and* $\sup |X_n| \to 0$. *Then the subspace*

$$X \equiv \{x \in H^r : x_n \in X_n \text{ for all } n\}$$

of H^r is totally bounded. If, in addition, each set X_n is compact, then X is compact.

Proof. It follows from Lemma 3.3 that X is located in H^r, from Lemma 3.8 that it is equisummable, and from Proposition 3.12 that it is bounded. Thus, by Theorem 3.9, X is totally bounded. When each set X_n is compact, then X is closed in H^r, and thus X is compact. □

We wish to relax the requirement that $0 \in X_n$ for all n. One method is the following; this result will be needed to construct the locally compact subspaces L^r in Theorem 5.6.

4.2. Theorem. *Let $\{X_n\}$ be a sequence of totally bounded subsets of $\mathbb{R}$, such that* $\sup |X_n| \to 0$. *Then the subspace*

$$X \equiv \{x \in H^r : x_n \in X_n \text{ whenever } x_n \neq 0\}$$

of H^r is totally bounded. If, in addition, each set X_n is compact, then also X is compact.

Proof. It follows from Lemma 3.8 that X is equisummable. Let $\varepsilon > 0$ and choose N so that $\sum_{n=N+1}^{\infty} |x_n| < \varepsilon/3$ for all $x \in X$. For each $n \leq N$, choose a finite $\varepsilon/3r$ approximation Y_n to X_n. Let Y be the set of all points $y \equiv (y_n)$ in H^r such that

(i) when $n \leq N$, then either $y_n = 0$ or $y_n \in Y_n$, and

(ii) when $n > N$, then $y_n = 0$.

Then Y is a finitely enumerable subset of X. Let $x \in X$ and partition $\{1, 2, \ldots, N\} = T \cup W$ so that $|x_n| < \varepsilon/3N$ for all $n \in T$, and $|x_n| > 0$ for all $n \in W$. It follows that $\operatorname{card} W \leq r$. For each $n \in W$, choose $y_n \in Y_n$ so that $|y_n - x_n| < \varepsilon/3r$, and define $y_n \equiv 0$ otherwise. Then the point $y \equiv (y_n)$ lies in Y, and $\varrho(x, y) = \sum_{n\in W} |y_n - x_n| + \sum_{n\in T} |x_n| + \sum_{n>N} |x_n| < r(\varepsilon/3r) + N(\varepsilon/3N) + \varepsilon/3 = \varepsilon$. This shows that X is totally bounded.

Now let each set X_n be compact, and let $\{x^k\}$ be a sequence in X with $x^k \to x$, where $x \equiv (x_n)$ is a point of H^r. If for some n we have $x_n \neq 0$, then for sufficiently large k we have $x_n^k \neq 0$, and thus $x_n^k \in X_n$; thus $x_n \in X_n$. Hence $x \in X$; this shows that X is closed in H^r. □

A slightly stronger condition produces a constructively smaller subspace which is *not* compact, as shown in the next example. (On the other hand, it was necessary to use a form of this stronger condition to obtain the finitely enumerable set Y in the last proof.)

4.3. Example. *The statement "If $\{X_n\}$ is a sequence of compact subsets of $\mathbb{R}$, such that* $\sup |X_n| \to 0$, *then the subspace*

$$X \equiv \{x \in H^r : \text{ for each } n, \text{ either } x_n = 0 \text{ or } x_n \in X_n\}$$

of H^r is compact" is nonconstructive; it implies LLPO.

Proof. It will suffice to consider the case $r = 1$. Given $a \in \mathbb{R}$, define $X_1 \equiv X_2 \equiv \{a\}$, and $X_n \equiv \{0\}$ for $n \geq 3$. Choose a decision sequence $\{\sigma_k\}$ so that $|a| < 1/k$ whenever $\sigma_k = 0$, and $|a| > 0$ whenever $\sigma_k = 1$. Define $x^k \equiv 0$ when $\sigma_k = 0$. When $\sigma_k = 1$, define $x^k \equiv (a, 0, \ldots)$ if $a > 0$, but $x^k \equiv (0, a, 0, \ldots)$ if $a < 0$. It is easy to verify that $\{x^k\}$ is a Cauchy sequence in X, and thus $x^k \to x$ for some point $x \equiv (x_n)$ in H^1. By hypothesis, $x \in X$; thus either $x_1 = 0$ or $x_1 = a$, and also either $x_2 = 0$ or $x_2 = a$. If $x_1 = 0$, then clearly $a \leq 0$, for $a > 0$ would imply $\sigma_k = 1$ eventually, and then $x_1 = a$. Similarly, if $x_2 = 0$, then $a \geq 0$. Finally, if $x_1 = x_2 = a$, then $a = 0$, for $a \neq 0$ would contradict the fact that $x \in H^1$. Thus LLPO obtains; this shows that X is not constructively complete. (The subspace X is totally bounded, as may be shown by the method used for the first part of Theorem 4.2.) □

Using Theorem 4.2 we can now return to the defining condition used in Theorem 4.1, while partially relaxing the condition that $0 \in X_n$ for all n. For this result, we will need *Bishop's Theorem*: *If X is a compact metric space, and $g : X \to \mathbb{R}$ is uniformly continuous, then the subspace $\{x \in X : g(x) \leq \alpha\}$ is compact for all but countably many real numbers $\alpha > \inf g$.* See [1, pp. 101-102], or [2, pp. 98-99].

4.4. Theorem. *Let $\{X_n\}$ be a sequence of compact subsets of $\mathbb{R}$, such that $\sup |X_n| \to 0$, and such that there is a finite set P of at most r positive integers, with $0 \in X_n$ for all $n \notin P$, and $\varrho(0, X_n) > 0$ for all $n \in P$. Then the subspace*

$$X \equiv \{x \in H^r : x_n \in X_n \textit{ for all } n\}$$

of H^r is compact.

Proof. Theorem 4.2 shows that the subspace

$$Y \equiv \{x \in H^r : x_n \in X_n \text{ whenever } x_n \neq 0\}$$

is compact. The function $f : Y \to \mathbb{R}$ defined by

$$f(x) = \prod_{n \in P} |x_n| \quad (x \in Y)$$

is uniformly continuous on Y. Define $\delta \equiv \prod_{n \in P} \varrho(0, X_n)$; it is then easily seen that $X = \{x \in Y : f(x) \geq \delta\} = \{x \in Y : f(x) > 0\}$. By Bishop's Theorem, there exists a real number α, with $0 < \alpha < \delta$, such that $\{x \in Y : f(x) \geq \alpha\}$ is compact; thus X is compact. □

In the above theorem, the condition that $\varrho(0, X_n) > 0$ for all $n \in P$ may *not* be omitted, as will be shown in the next example. We must first show, in the lemma below, that the condition "$x_n \neq 0$" used in the definition of H^r may be weakened to "x_n is not 0". For this it will be convenient to use the following proposition, a generalization of [7, 12.10].

4.5. Proposition. *If S is the negation of some statement R, and T is any statement such that $\neg\neg T$ is true, and $T \Rightarrow S$, then S is true.*

Proof. Since $T \Rightarrow S$, we have $\neg S \Rightarrow \neg T$, and thus

$$\neg\neg T \Rightarrow \neg\neg S \Leftrightarrow \neg\neg\neg R \Leftrightarrow \neg R \Leftrightarrow S.$$ □

4.6. Lemma. *For any point $x \equiv (x_n)$ in H, the following are equivalent.*

(i) $x \in H^r$.

(ii) *Whenever P is a finite set of r positive integers with x_n not 0 for all $n \in P$, then $x_m = 0$ for all $m \notin P$.*

(iii) *Whenever P is a finite set of positive integers with x_n not 0 for all $n \in P$, then* card $P \leq r$.

Proof. Let $x \in H^r$, let P be as specified in (ii), and let $m \notin P$. Define $c \equiv \prod_{n \in P} |x_n|$; then, by [7, 3.8], c is not 0. By the preceding proposition, to prove $x_m = 0$ we may assume $c > 0$. Thus $|x_n| > 0$ for all $n \in P$, and it follows that $x_m = 0$. The converse is immediate, as is the equivalence of (iii). □

4.7. Example. *The statement "If $\{X_n\}$ is a sequence of compact subsets of $\mathbb{R}$, such that* $\sup |X_n| \to 0$, *and such that there is a finite set P of at most r positive integers, with $0 \in X_n$ for all $n \notin P$, then the subspace*

$$X \equiv \{x \in H^r : x_n \in X_n \text{ for all } n\}$$

of H^r is compact" is nonconstructive; it is equivalent to WLPO.

Proof. It will suffice to give the proof for H^1. We will establish WLPO (*The Weak Limited Principle of Omniscience*) in the following form: *For any real number $a \geq 0$, either $a = 0$, or it is contradictory that $a = 0$.* We may assume that $a < 1$. Define $X_1 \equiv [a, 1]$, and $X_n \equiv [0, 1/n]$ for all $n \geq 2$. By hypothesis, X is compact. Define $y \equiv (0, 1, 0, \ldots)$; either $\varrho(y, X) < 1$ or $\varrho(y, X) > 1/2$. In the first case, choose a point $x \equiv (x_n)$ in X with $\varrho(y, x) < 1$; it follows that $x_2 > 0$. Thus $x_1 = 0$ and $a = 0$. In the second case, where $\varrho(y, X) > 1/2$, the point $(0, 1/2, 0, \ldots)$ cannot lie in X, and thus a cannot be 0. Thus WLPO obtains; this shows that X is *not* located. (Note. The subspace X is bounded and equisummable.)

Now assume WLPO. It follows from Lemma 3.8 that X is equisummable. Let $\varepsilon > 0$ and choose N so that $\sum_{n=N+1}^{\infty} |x_n| < \varepsilon/3$ for all $x \in X$, and so that $N \geq n$ for all $n \in P$. Using WLPO, we find that for each $n \in P$, either $0 \in X_n$ or $0 \notin X_n$ (i.e., $\varrho(0, X_n)$ is not 0); thus we may assume that $0 \notin X_n$ for all $n \in P$. For each $n \leq N$, choose a finite $\varepsilon/3r$ approximation Y_n to X_n, with $0 \in Y_n$ when $n \notin P$, and define

$$Y \equiv \{y \in H^r : y_n \in Y_n \text{ for } n \leq N, \text{ and } y_n = 0 \text{ for } n > N\}.$$

Then Y is a finite subset of X. Let $x \in X$ and partition $\{1, 2, \ldots, N\} = P \cup T \cup W$ so that $|x_n| < \varepsilon/3N$ for all $n \in T$, and $|x_n| > 0$ for all $n \in W$. It follows from the lemma that card $(P \cup W) \leq r$. For all $n \in P \cup W$, choose $y_n \in Y_n$ so that $|y_n - x_n| < \varepsilon/3r$, and define $y_n \equiv 0$ otherwise. Then the point $y \equiv (y_n)$ lies in Y, and $\varrho(x, y) < \varepsilon$. This shows that X is totally bounded. □

5. Locally compact subspaces

5.1. Definitions. In [1], [2], and [11] the term *locally compact* is used in various restricted senses. Here we use the term in its traditional sense: every point has a compact neighborhood. In the theory of metric spaces one is especially interested in those locally compact spaces which have metrizable one-point compactifications. These have been characterized constructively in [11]. Thus we say that a metric space X is *metrically locally compact* if it is the union of an increasing sequence of compact subsets, each of which is a uniform neighborhood of the preceding subset. (Classically, any neighborhood of a compact set is a uniform neighborhood, and a metric space is metrically locally compact if and only if it is locally compact and separable [3, p. 247].)

We first give some examples of subspaces of H which are not locally compact, and will then construct a large class of metrically locally compact subspaces of H.

5.2. Example. *Let $r \geq 1$. If x is any point of H^{r-1}, then x has no compact neighborhood in H^r. Thus H^r is not locally compact.*

Proof. If suffices to consider a point x of G^{r-1}. Let V be any neighborhood of x in H^r, choose $\varepsilon > 0$ so that the closed ε-sphere about x is contained in V, and choose a positive integer p so that $x_n = 0$ for all $n > p$. For each $k > p$, define $y_k^k \equiv \varepsilon$, and $y_n^k \equiv x_n$ for $n \neq k$. Then $y^k \equiv (y_n^k)$ is a point of V for each $k > p$, and this shows that V is not equisummable. It follows from Theorem 3.9 that V is not totally bounded. □

5.3. Definitions. For any $r \geq 1$, let U^r be the subspace of H consisting of all points with at least r nonzero terms. Let U be the subspace consisting of all points with infinitely many nonzero terms, and let H^* be the subspace consisting of all points with at most finitely many nonzero terms.

The method of Example 5.2 shows also that U^r, U, H^* and H are not locally compact.

The next proof requires *Bishop's Lemma*: *Let A be a complete located subset of a metric space X. If $x \in X$, and $\varrho(x,y) > 0$ for all $y \in A$, then $\varrho(x, A) > 0$.* See [7, 5.4].

5.4 Proposition.

(i) *For any $r \geq 1$, the subspace U^r is the metric complement $H - H^{r-1}$, and U^r is a dense open subset of H. Thus each space H^r is nowhere dense in H.*

(ii) *$U = \bigcap_{r=1}^{\infty} U_r$, and U is dense in H.*

(iii) *$H^* = \bigcup_{r=1}^{\infty} H^r$, and H^* is dense in H.*

Proof. Let $x \in U^r$, and choose a set Q of r positive integers such that $x_n \neq 0$ for all $n \in Q$. Let $y \in H^{r-1}$. Since $|y_n| > 0$ for all $n \in Q$ is not possible, we must have $|y_n| < |x_n|$ for some $n \in Q$. Thus $\varrho(x,y) > 0$. By Bishop's Lemma, $\varrho(x, H^{r-1}) > 0$, and thus $x \in H - H^{r-1}$.

Now let $x \in H - H^{r-1}$. Since $\varrho(x,0) > 0$, there is a positive integer n such that $x_n \neq 0$. If $r = 1$ this shows that $x \in U^1$. If $r > 1$ we use the term x_n to construct a point y of H^{r-1}. Since $\varrho(x,y) > 0$, there is a positive integer $m \neq n$ such that $x_m \neq 0$. Continuing, we obtain r nonzero terms of x; thus $x \in U^r$.

The remaining statements are easy to verify. □

5.5. Definition. Let L^r be the subspace of H consisting of all points with exactly r nonzero terms. Since it is clearly dense in H^r, the subspace L^r is located in H.

5.6. Theorem. *For any $r \geq 1$, the subspace L^r is the metric complement $H^r - H^{r-1}$, and L^r is metrically locally compact.*

Proof. $L^r = H^r \cap U^r = H^r - H^{r-1}$. For all positive integers $k \geq r$, and all n, define

$$X_n^k \equiv \begin{cases} \{a \in \mathbb{R} : 1/k \leq |a| \leq k\} & \text{when} \quad n \leq k, \\ \{0\} & \text{when} \quad n > k. \end{cases}$$

By Theorem 4.2, the subspaces

$$X^k \equiv \{x \in H^r : x_n \in X_n^k \text{ whenever } x_n \neq 0\}$$

are compact. By Bishop's Theorem, there exist real numbers α_k, with $1/(k+1) < \alpha_k < 1/k$, such that the subspaces

$$B_k \equiv \{x \in X^k : \varrho(x, H^{r-1}) \geq \alpha_k\}$$

are also compact. Clearly $L^r = \bigcup_{k=r}^{\infty} B_k$, and each set B_{k+1} is a uniform neighborhood of B_k in L^r. Thus L^r is metrically locally compact. □

5.7. Remarks. (1) For $r > 1$, the metric complement $H^r - \{0\}$ is not locally compact, as the method of Example 5.2 shows.

(2) The subspace L^r is not locally compact in the sense of [1] and [2], where a space is said to be locally compact if every bounded subset is contained in some compact subset. The set X of all points $x \equiv (x_n)$ in L^r with all $|x_n| \leq 1$ is bounded, by Proposition 3.12, but not equisummable, and thus, by Corollary 3.10, X is contained in no compact subset of L^r.

5.8. Example. *For any $r \geq 1$, the statement "H^r is σ-compact" is nonconstructive; it is equivalent to* LPO.

Proof. Let $\{a_k\}$ be a decision sequence. By hypothesis, $H^r = \bigcup_{k=1}^{\infty} X_k$, where each set X_k is compact. By Corollary 3.10, for each k there exist positive integers N_{kj} $(j \geq 1)$ such that $|x_n| < 1/j$ whenever $x \equiv (x_n)$ is a point of X_k and $n \geq N_{kj}$. We may assume that $\{N_{kj}\}$ is increasing in both indices. If $n = N_{kk}$ for some k, and $a_k < a_{k+1}$, we then define $x_n \equiv 1/k$; otherwise define $x_n \equiv 0$. This defines a point $x \equiv (x_n)$ of H^r. Choose k so that $x \in X_k$. Either $a_k = 1$ or $a_k = 0$. In the latter case, let $j \geq k$ and suppose $a_j < a_{j+1}$. Define $m \equiv N_{jj}$; thus $x_m = 1/j$. However, $m = N_{jj} \geq N_{kj}$, so $|x_m| < 1/j$, a contradiction. Thus all $a_j = 0$. Hence LPO obtains.

Now consider the converse. For any $k \geq 1$, Theorem 4.1 shows that the subspace

$$X^k \equiv \{x \in H^r : |x_n| \leq k \text{ when } n \leq k, \text{ and } x_n = 0 \text{ when } n > k\}$$

is compact. Clearly, $D^r = \bigcup_{k=1}^{\infty} X^k$, and thus D^r is σ-compact. Applying LPO, Example 2.7 shows that $D^r = H^r$; thus H^r is σ-compact. □

It follows that each space H^r is classically σ-compact, but not constructively. The space H, however, is not classically σ-compact. In fact, H is not σ-compact in a strong, affirmative sense:

5.9. Proposition. *The space H is not σ-compact, in the sense that if $\{X_k\}$ is any sequence of compact subsets of H, then there exists a point y of H such that $\varrho(y, X_k) > 0$ for all k.*

Proof. Using Theorem 3.11, choose, for each k, a positive integer N_k such that $\sum_{n=N_k}^{\infty} |x_n| < 1/2^k$ for all $x \in X_k$. We may assume that $\{N_k\}$ is increasing. Define $y_{N_k} \equiv 1/2^{k+1}$, and $y_n \equiv 0$ otherwise. Then $y \equiv (y_n)$ is a point of H. For any k, and any $x \in X_k$, we have $\sum_{n=N_k}^{\infty} |x_n| < \sum_{n=N_k}^{\infty} |y_n|$, and thus $\varrho(y, x) > 0$. It follows from Bishop's Lemma that $\varrho(y, X_k) > 0$. $\square$

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