Constructive Harmonic Conjugates

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Abstract

In the synthetic study of the real projective plane, harmonic conjugates have an essential role, with applications to projectivities, involutions, and polarity. The construction of a harmonic conjugate requires the selection of auxiliary elements; it must be verified, with an invariance theorem, that the result is independent of the choice of these auxiliary elements. A constructive proof of the invariance theorem is given here; the methods used follow principles put forth by Errett Bishop.

1 Introduction

The classical theory of the real projective plane is highly nonconstructive; it relies heavily, at nearly every turn, on the *Law of Excluded Middle.* For example, in classical treatises it is assumed that a given point either lies on a given line, or lies outside the line, although this assumption is constructively invalid. This is demonstrated in [7, Example 1.1], where it is shown that if taken in a strictly constructive sense, the assumption would lead to a solution of the Goldbach problem, and to solutions of many similar problems.

We follow the constructivist principles put forward by Errett Bishop [1]. Avoiding the *Law of Excluded Middle,* constructive mathematics is a generalization of classical mathematics, just as group theory, a generalization of abelian group theory, avoids the commutative law. Thus every result and proof obtained constructively is also classically valid. For the origins of modern constructivism, and the disengagement of mathematics from formal logic, see Bishop's "Constructivist Manifesto", Chapter 1 in [1] or [2]; further discussion and additional references will be found in [7].

A constructive real projective plane $\mathbb{P}$ is studied synthetically in [7]; topics include Desargues's Theorem, harmonic conjugates, projectivities, involutions, conics, Pascal's Theorem, and polarity. An analytic model, $\mathbb{P}^2(\mathbb{R})$, of the plane $\mathbb{P}$ is constructed in Euclidean space $\mathbb{R}^3$, and is used to prove the consistency of the axiom system; thus the plane $\mathbb{P}$ is referred to as a "real projective plane". Background information, references to related work in constructive geometry, and further properties of the plane $\mathbb{P}$ will be found in [7].

This paper is concerned with the *Invariance Theorem* for harmonic conjugates. The construction of the harmonic conjugate of a point requires the selection of auxiliary elements; it must be demonstrated that the result is uniquely determined, independent of the choice of these auxiliary elements.

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In an intuitionistic context, a harmonic conjugate construction was given by A. Heyting [6, Section 7]. However, the proof of the invariance theorem given there uses axioms for projective space; it does not apply to a projective plane constructed using only axioms for a plane. Moreover, the proof is incomplete; it applies only to points distinct from the base points. For applications of harmonic conjugates, e.g., to projectivities, involutions, and polarity, a complete proof is required.

A harmonic conjugate construction is given in [7], using only axioms for a plane; it applies uniformly to all points on the base line. However, the proof given there for the invariance theorem is incorrect; apart from the error, the proof is excessively complicated, and objectionable on several counts. The discovery of the error is due to Guillermo Calderón;[1] he later obtained a proof of the invariance theorem, using the method of [7], within a computer formalization of projective geometry [4].

Using a method which is simpler, more transparent, and more direct than the method used in [7], we will give a constructive proof of the invariance theorem in Section 3.

Following Bishop [1], we use no system of formal logic. Aside from ruling out use of the *Law of Excluded Middle,* no special rules are required. The constructive logic used here is known as *informal intuitionistic logic*; for a detailed treatment, see [3, Section 1.3].

The divergence of constructive logic from classical logic appears most sharply in the use of negation. The conditions $P \neq Q$, $l \neq m$, and $P \notin l$ do not have the connotation of negation, as they do classically. These relations acquire constructive properties determined by definitions and axioms, and become strong concepts; their negations are $P = Q$, $l = m$, and $P \in l$, which are weaker concepts. Thus the role of negation is in a sense reversed from the classical tradition.

The properties of the relations *equality* and *inequality*, as typically used in constructive studies, are seen most clearly in regard to real numbers. Certain concepts, such as $x = 0$, are relatively weak compared to stronger concepts, such as $x \neq 0$. The relation $x \neq 0$ requires the construction of an integer n such that $1/n < |x|$; it then follows that $x = 0$ is equivalent to $\neg(x \neq 0)$. The statement "$\neg(x = 0)$ *implies* $x \neq 0$" is constructively invalid. These facts motivate the selection of properties for concepts in constructive geometry.

In geometry, *a point lies outside a line,* $P \notin l$, is the stronger concept, while *a point lies on a line,* $P \in l$, is the weaker. On the constructive real metric plane $\mathbb{R}^2$, the geometrical and numerical concepts are directly related; $P \notin l$ if and only if $d(P, l) > 0$. In constructing the projective plane $\mathbb{P}$, we first specify conditions for the equality and inequality relations on the family of points and on the family of lines, and then say that the point P *lies outside the line* l, written $P \notin l$, if $P \neq Q$ for all points Q that lie on l [7, Definition 2.3]. The statement *"If* $\neg(P \notin l)$*, then* $P \in l$*"* is taken as an axiom, reflecting the constructive properties of the real numbers, while the statement *"If* $\neg(P \in l)$*, then* $P \notin l$*"* is constructively invalid.

A characteristic feature of constructivist method is meticulous use of the connective "or", the inclusive disjunction. To prove "A or B" constructively, it is required that either we prove A, or we prove B; it is not sufficient to prove $\neg(\neg A \text{ and } \neg B)$. An essential property, constructively valid for the real numbers, and assumed here

[1]e-mail from Montevideo, Uruguay, March 16, 2017. The error in the proof of [7, Theorem 4.7] is in the use of the conclusion of step (7) beyond that step, whereas it is valid only in relation to an assumption made in a previous step.

both for points and for lines, is the *cotransitivity property: "For any x, y, z, if $x \neq y$, then either $z \neq x$ or $z \neq y$".* Cotransitivity is very often the basis of a proof of an alternation. Also assumed for both points and lines is the *tightness property: "If $\neg(x \neq y)$, then $x = y$";* this reflects the property of real numbers noted above.

2 Preliminaries

Axioms, definitions, and results are given in [7] for the constructive real projective plane $\mathbb{P}$. One axiom has a preëminent standing in the axiom system; it is indispensable for virtually all constructive proofs involving the plane $\mathbb{P}$.

> **Axiom C7.** [7, Section 2] If l and m are distinct lines, and P is a point such that $P \neq l \cdot m$, then either $P \notin l$ or $P \notin m$.

This axiom may be viewed as a strongly-worded constructive form of the classical statement that the point common to two distinct lines is *unique.* If the classical statement has the form *"If l and m are distinct lines, with a common point denoted $l \cdot m$, and a point P is such that $P \in l$ and $P \in m$, then $P = l \cdot m$",* then the contrapositive of this statement, using classical logic, would be Axiom C7.

Heyting and van Dalen have used a variation of Axiom C7.[2] Paraphrased to fit the present context, this variation states: *If l and m are distinct lines, and P is a point such that $P \neq l \cdot m$, and $P \in l$, then $P \notin m$.* This is a seemingly weaker version of Axiom C7; however, Axiom C7 is easily derived from the weaker version.[3]

The proof of the invariance theorem requires Desargues's Theorem and its converse. The following definition includes explicit details which are required for constructive applications of Desargues's Theorem.

Definition 2.1. Two triangles are *distinct* if corresponding vertices are distinct and corresponding sides are distinct.[4]

Distinct triangles are said to be *perspective from the center O* if the lines joining corresponding vertices are concurrent at the point O, and O lies outside each of the six sides.

Distinct triangles are said to be *perspective from the axis l* if the points of intersection of corresponding sides are collinear on the line l, and each of the six vertices lies outside l.

Desargues's Theorem is adopted as an axiom, and then used to prove the converse.

Axiom D. Desargues's Theorem. *If distinct triangles are perspective from a center, then they are perspective from an axis.*

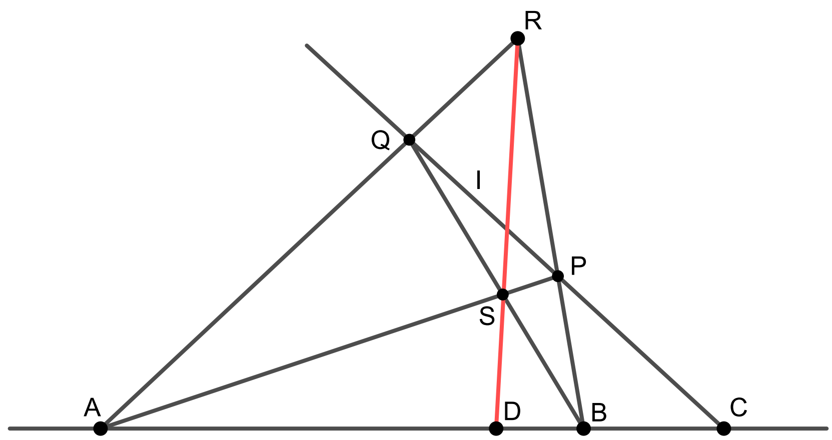
Theorem 2.2. [7, Theorem 3.2] *If distinct triangles are perspective from an axis, then they are perspective from a center.*

[2] Heyting's Axiom VI [6]; van Dalen's Lemma 3(f), obtained using his axiom Ax5 [5].

[3] Set $Q = l \cdot m$; by cotransitivity for lines, either $PQ \neq l$ or $PQ \neq m$, and thus either $P \notin l$ or $P \notin m$. This answers a question raised at [7, page 26, Note 4].

[4] It is then easily shown that the lines joining corresponding vertices are distinct, and the points of intersection of corresponding sides are distinct.

Although harmonic conjugates are often defined classically using quadrangles or triangles, we use a less problematic definition; it applies to every point on the line, including the base points.



Definition 2.3. [7, Definition 4.1] Let A and B be any distinct points. Given any point C on the line AB, select a line l that passes through C and is distinct from AB, and select a point R that lies outside each of the lines AB and l. Set $P = BR \cdot l$, $Q = AR \cdot l$, and $S = AP \cdot BQ$. Pending verification in Theorem 3.2, the point $D = AB \cdot RS$ will be called *the harmonic conjugate of C with respect to the points A and B*; we write $D = h(A, B; C)$.

The following preliminary results will be required for the proof of the invariance theorem.

Lemma 2.4. [7, Lemma 4.2] *In Definition 2.3 for the construction of a harmonic conjugate,*

(a) $P \neq A$, $Q \neq B$, $P \neq Q$.
(b) $P \notin AR$, $Q \notin BR$, $A \notin BR$, $B \notin AR$.
(c) $AR \neq BR$, $AP \neq AR$, $AP \neq BR$, $BQ \neq BR$, $BQ \neq AR$.

Lemma 2.5. [7, Lemma 4.4] *In Definition 2.3, $h(A, B; A) = A$ and $h(A, B; B) = B$, for any selection of the auxiliary elements l and R.*

Lemma 2.6. [7, Lemma 4.5] *In Definition 2.3 for the construction of a harmonic conjugate,*

(a) *If $C \neq A$, then $Q \notin AB$, $Q \neq S$, $S \neq A$, and $D \neq A$.*
(b) *If $C \neq B$, then $P \notin AB$, $P \neq S$, $S \neq B$, and $D \neq B$.*

Lemma 2.7. [7, Lemma 4.6] *In Definition 2.3 for the construction of a harmonic conjugate, let the point C be distinct from each base point; i.e., $C \neq A$ and $C \neq B$. Then the four points P, Q, R, S are distinct and lie outside the base line AB, and each subset of three points is noncollinear.*

3 The invariance theorem

In proving the invariance theorem, we consider first a situation in which the configuration allows application of Desargues's Theorem and its converse.

Lemma 3.1. *In Definition 2.3, let auxiliary element selections* (l, R) *and* (l', R') *be used to construct harmonic conjugates* D *and* D' *of the point* C*. If the point* C *is distinct from each base point; i.e.,* $C \neq A$ *and* $C \neq B$*, and*

$$AR' \neq AR,\ BR' \neq BR,\ \text{and}\ l' \neq l,\ l' \neq CP_1,\ l' \neq CQ_1,$$

where $P_1 = AP \cdot BR'$ *and* $Q_1 = BQ \cdot AR'$*, then* $D = D'$*.*

Proof. We first check the validity of the conditions. Since $C \neq A$ and $C \neq B$, the results of Lemma 2.7 will apply to the points P, Q, R, S, and also to the points P', Q', R', S'. Since $R' \notin AB$, we have $AB \neq BR'$. Since $A \neq B = AB \cdot BR'$, it follows from Axiom C7 that $A \notin BR'$, and thus $AP \neq BR'$. By symmetry, $BQ \neq AR'$. Thus the definitions of P_1 and Q_1 are valid. Also, we see that $A \neq P_1$ and $B \neq Q_1$. Since $P \notin AB$, we have $AB \neq AP$. Since $P_1 \neq A = AB \cdot AP$, it follows that $P_1 \notin AB$, and thus $P_1 \neq C$. Similarly, $Q_1 \neq C$. This shows that the conditions specified for l' are meaningful.

(1) Suppose that $D \neq D'$.

(2) Since $R \neq A = AR \cdot AR'$, it follows that $R \notin AR'$, and thus $R \neq R'$. From the conditions $AR' \neq AR$ and $BR' \neq BR$, we see that $QR \neq Q'R'$ and $PR \neq P'R'$. By Lemma 2.4(a), we see that $l = PQ$ and $l' = P'Q'$; thus $PQ \neq P'Q'$. Since $D \neq D' = AB \cdot R'S'$, it follows that $D \notin R'S'$, and thus $RS \neq R'S'$. Since $P' \neq C = l' \cdot CP_1$, it follows that $P' \notin CP_1$, and thus $P' \neq P_1$. Since $P' \neq P_1 = AP \cdot BR'$, it follows that $P' \notin AP$, and thus $P \neq P'$. Also, $AP \neq AP'$; i.e., $PS \neq P'S'$.[5] By symmetry, we have $Q' \notin BQ$, $Q \neq Q'$, and $QS \neq Q'S'$.[6] Since $S \neq A = AP \cdot AP'$, it follows that $S \notin AP'$, and thus $S \neq S'$.

The above, together with Lemma 2.7, shows that the quadrangles $PQRS$ and $P'Q'R'S'$ have distinct corresponding vertices and distinct corresponding sides, that the corresponding contained triangles are distinct, and that each of the eight vertices lies outside the line AB.

(3) The triangles PQR and $P'Q'R'$ have corresponding sides that meet at points $QR \cdot Q'R' = A$, $PR \cdot P'R' = B$, and $PQ \cdot P'Q' = l \cdot l' = C$; thus they are perspective from the axis AB. By the converse to Desargues's Theorem, the triangles are perspective from a center; setting $O = PP' \cdot QQ'$, it follows that $O \in RR'$. This also shows that O lies outside each of the six sides of these triangles, and that $O \neq R$ and $O \neq R'$.

(4) The triangles PQS and $P'Q'S'$ are also perspective from the axis AB; thus they are perspective from the center O, and it follows that $O \in SS'$. Also, O lies outside each of the six sides of these triangles, $O \neq S$, and $O \neq S'$.

(5) To apply Desargues's Theorem to the triangles PRS and $P'R'S'$, all the required distinctness conditions have been verified above, except for one; it remains to be shown that the point O lies outside each of the sides RS and $R'S'$.

[5] As for the necessity of taking the points P_1, Q_1 into account, note that if the line l' were to pass through the point P_1, then we would have $PS = P'S'$. Similarly, if l' were to pass through Q_1, then we would have $QS = Q'S'$. The proof in [7] does not take these points into account.

[6] Ibid.

It was shown at (2) that $RS \neq R'S'$; define $E = RS \cdot R'S'$. By cotransitivity, either $E \neq S$ or $E \neq R$. In the first case, since $S \neq E = RS \cdot R'S'$, it follows that $S \notin R'S'$, and thus $SS' \neq R'S'$. Since $O \neq S' = SS' \cdot R'S'$, it follows that $O \notin R'S'$. In the second case, since $R \neq E = RS \cdot R'S'$, it follows that $R \notin R'S'$, and thus $RR' \neq R'S'$. Since $O \neq R' = RR' \cdot R'S'$, it follows that $O \notin R'S'$. Thus in each case we obtain $O \notin R'S'$. Similarly, either $E \neq S'$ or $E \neq R'$, and by symmetry we find that $O \notin RS$.

(6) Now the triangles PRS and $P'R'S'$ are perspective from the center O. By Desargues's Theorem, these triangles are perspective from the axis $(PS \cdot P'S')(PR \cdot P'R') = AB$, and thus $RS \cdot R'S' \in AB$. Hence $E = D$ and $E = D'$, contradicting our assumption at (1); thus we have $\neg(D \neq D')$, and it follows from the tightness property for points that $D = D'$. □

Theorem 3.2. Invariance Theorem. *Let the projective plane $\mathbb{P}$ be such that at least eight distinct lines[7] pass through any given point. In Definition 2.3, let auxiliary element selections (l, R) and (l', R') be used to construct harmonic conjugates D and D' of the point C. Then $D = D'$; the harmonic conjugate construction is independent of the choice of auxiliary elements.*

Proof. We construct a third selection of auxiliary elements, and then utilize two applications of Lemma 3.1.

(1) Suppose that $D \neq D'$.

(2) By cotransitivity, either $A \neq D$ or $A \neq D'$; by symmetry, it will suffice to consider the first case. Since $A \neq D = AB \cdot RS$, it follows from Axiom C7 that $A \notin RS$, and thus $A \neq S$. Since $A \neq S = AP \cdot BQ$, it follows that $A \notin BQ$, and thus $A \neq Q$. Since $A \neq Q = AR \cdot l$, it follows that $A \notin l$, and thus $A \neq C$. Similarly, $B \neq C$. Thus the point C is distinct from each base point.

(3) Select a line m through the point A such that $m \neq AB$, $m \neq AR$, and $m \neq AR'$. Select a line n through the point B such that $n \neq AB$, $n \neq BR$, and $n \neq BR'$. Since $A \neq B = AB \cdot n$, it follows that $A \notin n$, and thus $m \neq n$. Define $R'' = m \cdot n$; since $A \notin n$, we have $A \neq R''$. Since $R'' \neq A = AB \cdot m$, it follows that $R'' \notin AB$.

Since $AR'' = m$, we have $AR'' \neq AR$, and $AR'' \neq AR'$; by symmetry, $BR'' \neq BR$ and $BR'' \neq BR'$. As verified in Lemma 3.1, we may define the points $P_1 = AP \cdot BR''$, $Q_1 = BQ \cdot AR''$, $P_2 = AP' \cdot BR''$, $Q_2 = BQ' \cdot AR''$, and note that these four points are distinct from the point C.

Now select a line l'' through the point C such that $l'' \neq CR''$; $l'' \neq l$, $l'' \neq CP_1$, $l'' \neq CQ_1$; and $l'' \neq l'$, $l'' \neq CP_2$, $l'' \neq CQ_2$. Since $R'' \neq C = CR'' \cdot l''$, it follows that $R'' \notin l''$.

(4) The auxiliary element selection (l'', R'') satisfies the conditions for Definition 2.3, resulting in a harmonic conjugate D''. This third selection (l'', R'') also satisfies the conditions of Lemma 3.1, relating it to each of the selections (l, R) and (l', R'). Two applications of Lemma 3.1 now show that $D'' = D$ and $D'' = D'$, contradicting our assumption at (1); thus we have $\neg(D \neq D')$, and it follows that $D = D'$. □

[7]In [7, Section 5], Axiom E stipulates that at least six distinct lines pass through any given point.

The harmonic conjugates constructed here can now be related to the traditional quadrangle configuration.[8]

Corollary 3.3. [7, Corollary 4.8] *Let A, B, C, D be collinear points, with $A \neq B$, and C distinct from both points A and B. Then $D = h(A, B; C)$ if and only if there exists a quadrangle $PQRS$, with vertices outside the line AB, such that $A = PS \cdot QR$, $B = PR \cdot QS$, $C \in PQ$, and $D \in RS$.*

Additional results concerning constructive harmonic conjugates, including applications to projectivities, involutions, and polarity, will be found in [7].

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[8] See, e.g., [8, Chapter IV].