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Open Subspaces of Locally Compact Metric Spaces

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Abstract

Although classically every open subspace of a locally compact space is also locally compact, constructively this is not generally true. This paper provides a locally compact remetrization for an open set in a compact metric space and constructs a one-point compactification.
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1. Introduction

In the constructive development of analysis, locally compact metric spaces play an important role. For example, constructive measure theory ([1], [2]) is developed mainly in this setting. A special limited form of local compactness is required for these purposes; a metric space is said to be *locally compact* if every bounded subset is contained in a compact subset. (A more general form of constructive local compactness may be found in [7], but will not be considered here.)

Under this definition many spaces, including an open disk in the plane, must be remetrized in order to become locally compact. A general method is desirable. A *located* subset F of a metric space X is a set for which the distance $\varrho(x, F)$ to any point x of X exists constructively. The *metric complement* $X - F$ of F is the subspace of X consisting of all points situated at a positive distance from F. (These and other basic constructive concepts may be found in [1] or [2].) For example, the open unit disk is the metric complement $U = X - F$ of the unit circle F in the compact disk X. The class of metric complements may be considered to be the open sets of constructive significance, and the results here are restricted to these. We give a construction to remetrize the metric complement U of any located set F in a compact metric space so that U is locally compact, and we show that F may be collapsed to a point ω in such a way that the resulting space X_F yields a one-point compactification of U, in which ω is the point at infinity.

2. The collapsing construction

Classically, any set may be used in this construction, but for constructive purposes a located set is required. This construction has also been used previously, for different purposes see [3, 4].

Theorem 1. *Let X be a metric space with metric ϱ, and let F be a located subset of X. Then*

$$d(x,y) \equiv \varrho(x,y) \wedge [\varrho(x,F) + \varrho(y,F)]$$

defines a pseudometric on X with the following properties:

(a) *If x and y both lie in the closure $\overline{F}$ of F, then $d(x,y) = 0$.*

(b) *If $y \in \overline{F}$, then for any $x \in X$, $d(x,y) = \varrho(x,F)$.*

Proof. Let $x, y, z \in X$ and let $\varepsilon > 0$. From the definition of $d(x,z)$, either

$$(1) \qquad \varrho(x,z) < d(x,z) + \varepsilon/2,$$

or

$$(2) \qquad \varrho(x,F) + \varrho(z,F) < d(x,z) + \varepsilon/2.$$

Similarly, either

$$(a) \qquad \varrho(z,y) < d(z,y) + \varepsilon/2,$$

or

$$(b) \qquad \varrho(z,F) + \varrho(y,F) < d(z,y) + \varepsilon/2.$$

In case (1a) we have

$$d(x,y) \le \varrho(x,y) \le \varrho(x,z) + \varrho(z,y) < d(x,z) + d(z,y) + \varepsilon.$$

In case (1b) we have

$$\begin{aligned} d(x,y) &\le \varrho(x,F) + \varrho(y,F) \le \varrho(x,z) + \varrho(z,F) + \varrho(y,F) \\ &< d(x,z) + d(z,y) + \varepsilon, \end{aligned}$$

and case (2a) is similar. Finally, in case (2b),

$$\begin{aligned} d(x,y) &\le \varrho(x,F) + \varrho(y,F) \le \varrho(x,F) + 2\varrho(z,F) + \varrho(y,F) \\ &< d(x,z) + d(z,y) + \varepsilon. \end{aligned}$$

It follows that $d(x,y) \le d(x,z) + d(z,y)$. The properties listed follow easily.

Definition. The set consisting of the elements of X with equality relation defined by $x \approx y$ if $d(x,y) = 0$, will be called the *collapsed set* of X by F, denoted by X_F. The set X_F with the metric d will be called the *collapsed space* of X by F.

Theorem 2. *Let (X, ϱ) be a metric space, let F be a located subset of X, let $X_0 \equiv X - F$, and let $p \in F$.*

(a) *The natural mapping $i : X \longrightarrow X_F$ is uniformly continuous.*

(b) *Restricted to X_0, the mapping i is an injection into X_F.*

(c) $i(X_0) = X_F - \{p\}$.

(d) *The inverse $j : i(X_0) \longrightarrow X_0$ is uniformly continuous on each compact subset of $i(X_0)$.*

(e) *If the space X is complete, and if $(Y, \tilde{d})$ is the completion of (X_F, d), then $X_F - \{p\} = Y - \{p\}$.*

Proof. (a), (b) and (c) follow directly from the definition of the metric d and Theorem 1.

(d). Let K be a compact subset of $X_F - \{p\}$. Using [6, Lemma 5.3], construct a point y in K so that $d(p,y) \leq 2d(p,K)$. This shows that K is bounded away from p with respect to d, and thus also bounded away from F with respect to ϱ. It follows that j is uniformly continuous on K.

(e). Let $y \in Y - \{p\}$. Then $\tilde{d}(y,p) > 0$, so there exists a sequence $\{x_n\}$ in X_F, bounded away from p with respect to d, with $x_n \to y$ with respect to $\tilde{d}$. Since $\{x_n\}$ is Cauchy in d, and bounded away from F with respect to ϱ, it is Cauchy in ϱ. Thus there exists $x \in X$ such that $x_n \to x$ with respect to ϱ. It follows that $x_n \to x$ with respect to d also, and thus $y = x$.

3. Remetrization of metric complements

The following generalizes a construction due to D. Bridges [2].

Theorem 3. *Let X_0 be the metric complement of a located set F in a compact metric space (X,ϱ), let X_0 be nonvoid, and fix a point ω in F. Then X_0 is locally compact in the metric*

$$\varrho_0(x,y) \equiv \{\varrho(x,y) \wedge [\varrho(x,F) + \varrho(y,F)]\} + |1/\varrho(x,F) - 1/\varrho(y,F)|,$$

and the completion Y of X_F is a one-point compactification of (X_0, ϱ_0), with point at infinity ω.

Proof. Since X is totally bounded, so is (X_F, d), and thus $(Y, \tilde{d})$ is compact. Proposition 4.6.9 of [2] applies directly to $Y - \{\omega\}$; it is locally compact relative to the new metric $\tilde{d}_0$, and Y is a one-point compactification, with point at infinity ω. Since X is complete, $Y - \{\omega\}$ is merely $X_F - \{\omega\}$, which is thus locally compact in the metric

$$d_0(x,y) \equiv d(x,y) + |1/d(x,\omega) - 1/d(y,\omega)|.$$

The correspondence between X_0 and $X_F - \{\omega\}$ induces the metric ϱ_0, with respect to which X_0 is locally compact, with Y a one-point compactification.

4. Examples

a) Let X be the closed unit interval $[0,1]$, and let F be the set $\{0,1\}$ of endpoints. Then X_F may be thought of as a loop. The open interval $X_0 = (0,1)$ is locally compact in the metric ϱ_0 and the circle is a one-point compactification. However, the loop X_F itself is constructively only a *non*complete subspace of the circle. There are Cauchy sequences on the loop which have no limit because we cannot tell on which side of the joint they cluster. Points in X_F are born of points in $[0,1]$, and remember their parentage.

b) On the real line $\mathbb{R}$, [5] characterizes the metric complement U of a located set F as a countable union of disjoint open intervals. When U is bounded, it is locally compact under the metric ϱ_0. The distances $\varrho(x,F)$ used in calculating ϱ_0 are simply distances to the endpoints of the intervals. The one-point compactification is formed by identifying the points of F, and forming the completion. When U resolves into two finite open intervals, $\mathbb{R}_F$ may be realized as a curve in the form of the symbol ∞. With more components, there are more loops; the one-point compactification is the completion, and the point at infinity is the junction.

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