ROUND z-FILTERS AND ROUND SUBSETS OF βX

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ABSTRACT

A characterization of remote points in $\beta \mathbf{R}$ is given by means of prime z-filters and is generalized to z-filters on arbitrary completely regular Hausdorff spaces X and to subsets of βX .

1. Introduction. An interesting topological feature of the Stone-Čech compactification $\beta \mathbf{R}$ of the real line is the existence of a *remote point* in $\beta \mathbf{R}$, i.e., a point not in the closure of any discrete subset of **R**. Other characterizations and properties of remote points are given in [1], [6], [7] and [8]. The existence of remote points in $\beta \mathbf{R}$ has been demonstrated, assuming the continuum hypothesis, in [1] and in [8]. The present paper was motivated by the following characterization of remote points.

THEOREM. A prime z-filter \mathcal{P} on \mathbf{R} has the property that for every $Z \in \mathcal{P}$ there exists $W \in \mathcal{P}$ such that $W \subseteq$ int Z if and only if $\mathcal{P} = \mathcal{M}^p$ for some remote point p in $\beta \mathbf{R}$.

The proof will be given in Section 2. For terminology and notation see [2]; we use \mathcal{M}^p and \mathcal{O}^p to denote the z-filters $\mathbb{Z}[\mathcal{M}^p]$ and $\mathbb{Z}[\mathcal{O}^p]$. We generalize the property of the theorem to an arbitrary completely regular Hausdorff space X as follows.

DEFINITION. A z-filter \mathscr{F} on X will be called *round* if for every $Z \in \mathscr{F}$ there is $W \in \mathscr{F}$ and a cozero-set S in X such that $W \subseteq S \subseteq Z$.

The term "round" is borrowed from [5]. It is used in [10] in connection with proximity spaces. In fact, round z-filters are just the intersections with Z(X) of the round filters on X with respect to the proximity induced by βX . In the case of the real line, every open set is a cozero-set, so our condition reduces to that of the theorem above.

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The basic properties of round z-filters will be obtained in Section 3. Round subsets of βX are introduced in Section 4; they are related to an interesting class of round z-filters. In Section 5 we find that every G_{δ} in βX that does not meet X is a round subset of βX , and we relate this result to the family of continuous functions with compact support.

2. Round prime z-filters on the real line. In the proof of the theorem stated in the introduction we shall need some results from [6] which we restate in the first two lemmas below.

LEMMA 2.1. A prime z-filter on \mathbf{R} is minimal if and only if each of its members has nonempty interior. [6, Theorem 8.1].

LEMMA 2.2. A point p in $\beta \mathbf{R}$ is a remote point if and only if the z-ultrafilter \mathcal{M}^p is a minimal prime z-filter. (Part of [6, Theorem 11.2].)

Lemma 2.1 shows that a nonminimal prime z-filter on **R** has a nowhere dense member. We shall also need the following stronger result. We use the symbol \subset to denote proper inclusion.

LEMMA 2.3. If \mathscr{P} and \mathscr{Q} are prime z-filters on **R** with $\mathscr{P} \subset \mathscr{Q}$, then there exists $Z \in \mathscr{P}$ such that bdry $Z \in \mathscr{Q}$.

Proof. Since \mathscr{Q} is nonminimal, we may choose a nowhere dense $F \in \mathscr{Q}$. By enlarging F slightly if necessary (e.g., by adjoining the integers), we may assume that all the components of $\mathbf{R} - F$ are bounded. Let U be the union of all the left open thirds of these components, and V of the right. Since cl U contains the set of left endpoints, which is dense in F, we have $F \subseteq clU$, and similarly for V. Let Z and W be the complements of U and V in \mathbf{R} . Since U and V are disjoint, we have $Z \cup W = \mathbf{R}$. Say $Z \in \mathscr{P}$. Since $F \subseteq cl(\mathbf{R} - Z)$, we have $F \cap Z \subseteq$ bdry Z and hence bdry $Z \in \mathscr{Q}$.

The proof also shows that on the real line every nowhere dense set is regularly nowhere dense in the sense of [3]. (The author wishes to thank Jack R. Porter for this remark and also for a simplification included in the above proof.).

THEOREM 2.4. A prime z-filter \mathcal{P} on **R** is round if and only if $\mathcal{P} = \mathcal{M}^p$ for some remote point p in $\beta \mathbf{R}$.

Proof. Let \mathscr{P} be round. Suppose $\mathscr{P} \subset \mathscr{Q}$ for some (prime) z-filter \mathscr{Q} . Choose $Z \in \mathscr{P}$ such that bdry $Z \in \mathscr{Q}$ and choose $W \in \mathscr{P}$ such that $W \subseteq \text{int } Z$. Then \mathscr{Q} contains

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the disjoint sets W and bdry Z, which is absurd. Hence \mathcal{P} is a z-ultrafilter and $\mathcal{P} = \mathcal{M}^p$ for some $p \in \beta \mathbf{R}$. Since \mathcal{P} is clearly minimal, p is a remote point.

Conversely, let $\mathscr{P} = \mathscr{M}^p$ for a remote point p. Let $Z \in \mathscr{P}$. Since \mathscr{P} is minimal, bdry $Z \notin \mathscr{P}$. Thus, since \mathscr{P} is a z-ultrafilter, there is $F \in \mathscr{P}$ with $F \cap$ bdry $Z = \emptyset$. Hence $W = F \cap Z \subseteq \text{int } Z$.

3. Round z-filters. In this section we develop the basic properties of round z-filters. An auxiliary result is the determination in Theorem 3.5 of the z-filters on X with a prescribed set of cluster points in βX .

DEFINITION. For any z-filter \mathscr{F} on X, we denote by \mathscr{F}^0 the family of all zerosets Z in X such that there is a member W of \mathscr{F} and a cozero-set S in X such that $W \subseteq S \subseteq Z$.

LEMMA 3.1. For any z-filter \mathcal{F} , the family \mathcal{F}^0 is a round z-filter; \mathcal{F} is round if and only if $\mathcal{F} = \mathcal{F}^0$.

Proof. It is easy to verify that \mathscr{F}^0 is a z-filter. Let $Z \in \mathscr{F}^0$ and choose $W \in \mathscr{F}$ and a cozero-set S such that $W \subseteq S \subseteq Z$. Since W and X - S are disjoint zero-sets, they are completely separated. Hence by [2, 1.15(a)] there is a zero-set F and a cozero-set T such that $W \subseteq T \subseteq F \subseteq S \subseteq Z$. Hence $F \in \mathscr{F}^0$; this shows that \mathscr{F}^0 is round. The last statement is clear.

We adapt the notation of [2, 70] to z-filters as follows.

DEFINITION. For any z-filter \mathscr{F} on X, we denote by $\theta(\mathscr{F})$ the set of all cluster points of \mathscr{F} in βX . That is,

$$\theta(\mathscr{F}) = \bigcap_{Z \in \mathscr{F}} \mathrm{cl}_{\beta X} Z = \{ p \in \beta X \colon \mathscr{F} \subseteq \mathscr{M}^p \}.$$

Every nonempty closed subset A of βX is of the form $\theta(\mathscr{F})$; for example, we may take $\mathscr{F} = \{Z \in \mathbb{Z}(X) : A \subseteq cl_{\beta X}Z\} = \bigcap_{p \in A} \mathscr{M}^p$.

The following useful result arises in the proof of [2, 70.2].

LEMMA 3.2. If \mathscr{F} is any z-filter on X and Z is a zero-set in X such that $cl_{\beta X}Z$ is a neighborhood of $\theta(\mathscr{F})$, then there exists $W \in \mathscr{F}$ such that $cl_{\beta X}Z$ is a neighborhood of $cl_{\beta X}W$.

Proof. The family of all sets of the form $\operatorname{cl}_{\beta X} W$, where $W \in \mathscr{F}$, is closed under finite intersection and has intersection $\theta(\mathscr{F})$. Since $\theta(\mathscr{F}) \cap (\beta X - \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z) = \emptyset$, by compactness there is $W \in \mathscr{F}$ such that $\operatorname{cl}_{\beta X} W \cap (\beta X - \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z) = \emptyset$.

THEOREM 3.3. If \mathcal{F} is any z-filter on X, the following are equivalent. (a) \mathcal{F} is a round z-filter.

(b) For every $Z \in \mathscr{F}$ there is $W \in \mathscr{F}$ such that $cl_{\beta X}Z$ is a neighborhood of $cl_{\beta X}W$.

(c) For any $p \in \beta X$, $\mathcal{F} \subseteq \mathcal{M}^p$ implies $\mathcal{F} \subseteq \mathcal{O}^p$.

(d) For every $Z \in \mathcal{F}$, $cl_{\beta X} Z$ is a neighborhood of $\theta(\mathcal{F})$.

Proof. (a) is equivalent to (b) by [2, 7.14].

(b) implies (c). If $Z \in \mathscr{F}$ and W is chosen as in (b), then $W \in \mathscr{M}^p$; hence $p \in cl_{\beta X}W$, $cl_{\beta X}Z$ is a neighborhood of p, and $Z \in \mathcal{O}^p$.

(c) implies (d). If $p \in \theta(\mathscr{F})$, then $\mathscr{F} \subseteq \mathscr{M}^p$; hence $\mathscr{F} \subseteq \mathscr{O}^p$ and $cl_{\beta X}Z$ is a neighborhood of p.

(d) implies (b). Lemma 3.2.

We now determine which z-filters on X have a prescribed set of cluster points in βX . The lemma below extends [2, 7H.1]. The theorem generalizes [2, 7.13]; the necessity is [2, 70.2].

LEMMA 3.4. Any closed subset A of βX has a base of neighborhoods of the form $cl_{\beta X}Z$ with $Z \in \mathbb{Z}(X)$.

Proof. If U is any open neighborhood of A in βX , then A and $\beta X - U$ are completely separated. Hence there is a zero-set-neighborhood W of A in βX with $W \subseteq U$. Put $Z = W \cap X$. Then $Z \in \mathbb{Z}(X)$ and $cl_{\beta X}Z$ is clearly a neighborhood of A that is contained in U.

THEOREM 3.5. Let A be any closed subset of βX . For any z-filter \mathcal{F} on X, we have $\theta(\mathcal{F}) = A$ if and only if $\bigcap_{p \in A} \mathbb{O}^p \subseteq \mathcal{F} \subseteq \bigcap_{p \in A} \mathcal{M}^p$.

Proof. Let $\theta(\mathscr{F}) = A$. If $Z \in \bigcap_{p \in A} \mathcal{O}^p$ then $\operatorname{cl}_{\beta X} Z$ is a neighborhood of $\theta(\mathscr{F})$. Thus by Lemma 3.2 there is $W \in \mathscr{F}$ such that $\operatorname{cl}_{\beta X} Z$ contains $\operatorname{cl}_{\beta X} W$; hence $W \subseteq Z$ and so $Z \in \mathscr{F}$. Conversely, we clearly have $A \subseteq \theta(\mathscr{F})$. By Lemma 3.4, $\theta(\bigcap_{p \in A} \mathcal{O}^p) = A$ and hence $\theta(\mathscr{F}) \subseteq A$.

By means of the next two results we completely characterize round z-filters in terms of intersections of the z-filters \mathcal{O}^p .

THEOREM 3.6. For any z-filter \mathscr{F} on X, we have $\mathscr{F}^0 = \bigcap_{p \in \theta(\mathscr{F})} \mathscr{O}_p$.

Proof. $Z \in \mathscr{F}^0$ if and only if there is $W \in \mathscr{F}$ such that $W \subseteq S \subseteq Z$ for some cozero-set S in X; equivalently by [2, 7.14], such that $cl_{\beta X}Z$ is a neighborhood of $cl_{\beta X}W$. Also, $Z \in \bigcap_{p \in \theta(\mathscr{F})} \mathcal{O}^p$ if and only if $cl_{\beta X}Z$ is a neighborhood of $\theta(\mathscr{F})$; equivalently by Lemma 3.2, a neighborhood of $cl_{\beta X}W$ for some $W \in \mathscr{F}$.

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THEOREM 3.7. If \mathscr{F} is a round z-filter on X, then $\mathscr{F} = \bigcap_{p \in \theta(\mathscr{F})} \mathbb{O}^p$. Conversely, if A is any nonempty closed subset of βX , then $\bigcap_{p \in A} \mathbb{O}^p$ is a round z-filter; these are distinct for distinct closed sets A.

Proof. The first statement follows from Lemma 3.1 and the last theorem. Now let A be a nonempty closed subset of βX and put $\mathscr{F} = \bigcap_{p \in A} \mathcal{O}^p$. By Theorem 3.5, $\theta(\mathscr{F}) = A$ and hence \mathscr{F} is round by Theorem 3.3(d). The last statement also follows from Theorem 3.5.

COROLLARY 3.8. $A \to \bigcap_{p \in A} \mathcal{O}^p$ is a one-one order-reversing correspondence between the nonempty closed subsets of βX and the round z-filters on X.

COROLLARY 3.9. A round z-filter is a z-filter that is minimal with respect to its set of cluster points.

If \mathscr{P} is a prime z-filter, then $\theta(\mathscr{P})$ has just one point; hence every round prime filter is \mathscr{O}^p for some $p \in \beta X$. Since on the real line \mathscr{O}^p is prime if and only if p is a remote point in $\beta \mathbf{R}$ [6, Theorem 11.2], this provides another proof of Theorem 2.4. Alternatively, Theorem 3.3(c) could be used. However, Lemma 2.3 and the direct proof given in Section 2 are of independent interest and do not require the stronger result from [6] that if \mathscr{O}^p is prime then p is a remote point. In fact, Section 2 and Theorem 3.7 provide another proof of this result.

4. Round subsets of βX . Lemma 2.2 shows that p is a remote point in $\beta \mathbf{R}$ if and only if $\mathcal{M}^p = \mathcal{O}^p$, i.e., for any $Z \in \mathbf{Z}(\mathbf{R})$, if $cl_{\beta \mathbf{R}}Z$ contains p, then it is a neighborhood of p. In Section 2 we found the relation between remote points and round prime z-filters. We now generalize the above characterization of remote points, obtaining a class of subsets of βX which is related to a larger class of round z-filters.

DEFINITION. A subset A of βX will be called a round subset of βX if for any $Z \in \mathbb{Z}(X)$, if $cl_{\beta X}Z$ contains A, then it is a neighborhood of A.

We collect in the next theorem some immediate properties of round subsets.

THEOREM 4.1. Let $A \subseteq \beta X$.

(a) A is a round subset of βX if and only if $\bigcap_{p \in A} \mathscr{M}^p = \bigcap_{p \in A} \mathscr{O}^p$.

(b) If $cl_{\beta X}A$ is a round subset of βX , then A is also round.

(c) Every open subset of βX is round.

(d) Any union of round subsets of βX is also round.

THEOREM 4.2. For any nonempty closed subset A of βX , the following are equivalent.

- (a) A is a round subset of βX .
- (b) $\bigcap_{p \in A} \mathcal{M}^p$ is a round z-filter.
- (c) $\bigcap_{p \in A} \mathcal{M}^p = \bigcap_{p \in A} \mathcal{O}^p$.
- (d) There is a unique z-filter \mathscr{F} on X such that $\theta(\mathscr{F}) = A$.

Proof. (a) implies (b). We apply Theorem 3.3(d). Since A is closed, $\theta(\bigcap_{p \in A} \mathcal{M}^p) = A$. For every $Z \in \bigcap_{p \in A} \mathcal{M}^p$, $cl_{\beta X} Z$ contains A, and is hence a neighborhood of A. Thus $\bigcap_{p \in A} \mathcal{M}^p$ is a round z-filter.

(b) implies (c). Since $\theta(\bigcap_{p \in A} \mathcal{M}^p) = A$, this follows from Theorem 3.7.

(c) implies (a). Theorem 4.1(a).

(c) and (d) are equivalent by Theorem 3.5.

EXAMPLE 4.3. Let X be connected and let Z be any proper zero-set in X with nonempty interior. Let A be the interior of $cl_{\beta X}Z$ and let B be the closure of A. Since A is open, it is a round subset of βX . Since βX is connected, A is not closed. Hence $cl_{\beta X}Z$ contains B but is not a neighborhood of B. Thus the closure of a round subset of βX need not be round; i.e., the converse of Theorem 4.1(b) is not true. We also have $\bigcap_{p \in A} \mathcal{O}^p \neq \bigcap_{p \in B} \mathcal{O}^p$, whereas the corresponding z-filters using \mathcal{M}^p are equal for any $A \subseteq \beta X$ and any X. Furthermore, $\bigcap_{p \in A} \mathcal{M}^p$ is not round, since it is equal to $\bigcap_{p \in B} \mathcal{M}^p$ which is not round by Theorem 4.2. Thus the hypothesis that A is closed in Theorem 4.2((a) implies (b)) may not be removed. Since A is round, we have $\bigcap_{p \in A} \mathcal{M}^p = \bigcap_{p \in A} \mathcal{O}^p$ and hence we see that $\bigcap_{p \in A} \mathcal{O}^p$ is not round. Thus the hypothesis that A is closed in the second part of Theorem 3.7 may not be removed, even if A is round. Also, the intersection of a family of round z-filters need not be round.

5. Functions with compact support. Theorem 8.19 of [2] (see also [9]) shows that when X is realcompact, the intersection of all the free maximal ideals in C(X) is the family $C_{K}(X)$ of all functions with compact support. This property of a realcompact space may be restated in terms of round subsets of βX . For any space X, [2, 7E] shows that $C_{K}(X) = \bigcap_{p \in \beta X - X} O^{p}$. Thus the following is immediate.

THEOREM 5.1. The intersection of all the free maximal ideals in C(X) is the family $C_{K}(X)$ of all functions with compact support if and only if $\beta X - X$ is a round subset of βX .

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Thus [2, 8.19] says that if X is realcompact, then $\beta X - X$ is a round subset of βX . This will be generalized in Theorem 5.3 below. We first state a well-known lemma.

LEMMA 5.2. The following are equivalent.

(a) X is realcompact.

(b) $\beta X - X$ is a union of zero-sets in βX .

(c) $\beta X - X$ is a union of G_{δ} -sets in βX .

Proof. (a) implies (b). If $p \in \beta X - X$ then M^* contains a unit of C[2, 7.9(b)], i.e., there is $f \in C^*(X)$ such that $p \in \mathbb{Z}(f^\beta)$ and $\mathbb{Z}(f^\beta) \subseteq \beta X - X$.

(b) implies (c). Trivial.

(c) implies (a). The complement of a G_{δ} is an F_{σ} , hence (in the compact space βX) σ -compact and thus realcompact. Hence X is an intersection of realcompact subspaces of βX , and thus by Theorem 8.9 of [2] is itself realcompact.

THEOREM 5.3. For any space X, every G_{δ} in βX that does not meet X is a round subset of βX .

Proof. It follows from the lemma that if a subset of βX does not meet X and is a union of G_{δ} -sets, then it is a union of zero-sets. Thus, since the family of round subsets of βX is closed under arbitrary unions, it suffices to prove the theorem in the case of a zero-set W in βX that does not meet X. Let $W \subseteq cl_{\beta X} Z$ for some $Z \in \mathbb{Z}(X)$. Put $T = \beta X - W$, choose $f \in C(\beta X)$ so that $W = \mathbb{Z}(f)$, and put h = 1/f on T. Suppose W meets $cl_{\beta X}(X - Z)$. Then h, which is continuous on \mathbb{T} is unbounded on X - Z; thus X - Z contains a noncompact set S that is Cembedded and closed in T[2, 1.20]. Thus S is C-embedded in X and is thus completely separated in X from the zero-set Z [2, 1.18]; hence S and Z have disjoint closures in βX . Since S is closed in T but not compact we may choose $p \in cl_{\beta X}S - T$. Thus $p \in W$ but $p \notin cl_{\beta X}Z$, contradicting our assumption concerning Z. It follows that $W \cap cl_{\beta X}(X - Z) = \emptyset$. Thus $W \subseteq \beta X - cl_{\beta X}(X - Z) \subseteq cl_{\beta X}Z$ and $cl_{\beta X}Z$ is a neighborhood of W. Hence W is a round subset of βX .

COROLLARY 5.4. For any space X, $\beta X - vX$ is a round subset of βX .

Proof. Put Y = vX. Then Y is realcompact, so $\beta Y - Y$ is a union of G_{δ} -sets in $\beta Y = \beta X$. Since these G_{δ} -sets do not meet X, they are round subsets of βX , and hence their union $\beta X - vX$ is also a round subset of βX .

COROLLARY 5.5.[2, Theorem 8.19]. If X is realcompact, then $\beta X - X$ is a

round subset of βX ; i.e., the intersection of all the free maximal ideals in C(X) is the family $C_{K}(X)$ of all functions with compact support.

It is shown in [4, 3.9] that if X is a P-space, then the intersection of the free maximal ideals in C(X) is the family of functions with compact support. This result may be extended as follows.

THEOREM 5.6. X is a P-space if and only if every subset of βX is round.

Proof. X is a P-space if and only if $\mathcal{M}^p = \mathcal{O}^p$ for all $p \in \beta X$ [2, 7L.1]; the result thus follows from Theorem 4.1(a).

EXAMPLE 5.7. For any $p \in \beta \mathbf{R} - \mathbf{R}$, it is clear that $\{p\}$ is a round subset of $\beta \mathbf{R}$ if and only if p is a remote point in $\beta \mathbf{R}$. Under the continuum hypothesis, the set Γ of remote points in $\beta \mathbf{R}$ is dense in $\beta \mathbf{R} - \mathbf{R}$ ([1, 2.5] or [8, 5.4]). Hence every (round) open subset of $\beta \mathbf{R}$ that meets $\beta \mathbf{R} - \mathbf{R}$ contains a remote point, However, not every round subset of $\beta \mathbf{R}$ that meets $\beta \mathbf{R} - \mathbf{R}$ contains a remote point. For example, the set Δ of points in $\beta \mathbf{R} - \mathbf{R}$ that are not remote points in $\beta \mathbf{R}$ is also dense in $\beta \mathbf{R} - \mathbf{R}$ [1, 3.3]. By Corollary 5.5, $cl_{\beta \mathbf{R}} \Delta = \beta \mathbf{R} - \mathbf{R}$ is a round subset of $\beta \mathbf{R}$ that neets of $\beta \mathbf{R}$ but Δ contains no remote point.

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