A CONSTRUCTIVE VERSION OF THE SYLVESTER-GALLAI THEOREM

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Abstract. The Sylvester-Gallai Theorem, stated as a problem by James Joseph Sylvester in 1893, asserts that for any finite, noncollinear set of points on a plane, there exists a line passing through exactly two points of the set. First, it is shown that for the real plane $\mathbb{R}^2$ the theorem is constructively invalid. Then, a well-known classical proof is examined from a constructive standpoint, locating the nonconstructivities. Finally, a constructive version of the theorem is established for the plane $\mathbb{R}^2$; this reveals the hidden constructive content of the classical theorem. The constructive methods used are those proposed by Errett Bishop.[1]

1. Introduction

The Sylvester-Gallai Theorem states that for any finite, noncollinear set of points on a plane, there exists a line passing through exactly two points of the set.

The history of this problem is itself problematic. The notion, that Gallai was the first to prove the theorem, appears to stem from the submission of the problem to the *American Mathematical Monthly* in 1943 by P. Erdős [Erd43], while unaware of the 1893 statement of the problem by J. J. Sylvester [Syl93]. When a solution by R. Steinberg [Ste44] was published in the *Monthly* in 1944, an *Editorial Note* stated that Erdős had "enclosed with the problem an outline of Grünwald's [Gallai's] solution". This reference to a pre-1944 proof by Gallai, albeit an unpublished outline, appears to be the basis for the designation *Sylvester-Gallai Theorem.* This attribution recurs in a 1982 statement by Erdős [Erd82, p.208], "In 1933 . . . I told this problem to Gallai who very soon found an ingenious proof"; Gallai's proof, it seems, was not published. Along with Steinberg's 1944 solution, the *Monthly* noted that solutions were also received from R. C. Buck and N. E. Steenrod. The earliest known published proof of the theorem, appearing in 1941, is due to E. Melchior [Mel41]. In the present paper we accede to common usage, refraining from use of the designation *Sylvester-Melchior Theorem.*

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There have been many different versions and proofs for this theorem. V. Pambuccian [Pam09], conducts reverse analyses of three proofs of the theorem, leading to three different and incompatible axiom systems. J. von Plato [Pla05] shows that the theorem holds intuitionistically for sets of up to six points in a purely incidence-geometric setting, and for up to seven points in an ordered geometric setting. See also [KelMos58, Wil68, Cha70, Lin88, BorMos90, Chv04].

We determine the constructive content of this theorem for the real plane $\mathbb{R}^2$. First, we find that the theorem is constructively invalid. Then we examine L. M. Kelly's 1948 proof,[2] locating the nonconstructivities. Finally, adapting Kelly's method, adding an hypothesis, and using strictly constructive methods, we obtain a constructive version of the theorem.

2. Constructive methods

The modern constructivist program began with L. E. J. Brouwer (1881-1966) [Bro08]; recent work, using the strictest methods, follows the work of Errett Bishop (1928-1983). A large portion of analysis is constructivized by Bishop in *Foundations of Constructive Analysis* [B67]; this treatise also serves as a guide for constructive work in other fields. This variety of constructivism does not form a separate branch of mathematics, nor is it a branch of logic; it is intended as an enhanced approach for all of mathematics.

For the distinctive characteristics of Bishop-type constructivism, as opposed to intuitionism or recursive function theory, see [BR87]. Avoiding the *Law of Excluded Middle* (LEM), constructive mathematics is a generalization of classical mathematics, just as group theory, a generalization of abelian group theory, avoids the commutative law.[3]

The initial phase of this program involves the rebuilding of classical theories, using only constructive methods. The entire body of classical mathematics is viewed as a wellspring of theories waiting to be constructivized.

> Every theorem proved with [nonconstructive] methods presents a challenge: to find a constructive version, and to give it a constructive proof.
>
> \- Errett Bishop [B67, p. x]

To clarify the constructive methods used here, we give examples of familiar properties of the real numbers that are constructively *invalid*, and also properties that are constructively *valid*.[4]

The following classical properties of a real number α are constructively *invalid*:

(i) *Either* $\alpha < 0$ *or* $\alpha = 0$ *or* $\alpha > 0$.

(ii) *If* $\neg(\alpha = 0)$, *then* $\alpha \neq 0$.

Bishop constructs the real numbers using Cauchy sequences. Constructively, the relation $\alpha \neq 0$ does not refer to negation, but is given a strong affirmative definition; one must construct an integer n such that $1/n < |\alpha|$.

Among the resulting constructively *valid* properties of the reals are the following:

(a) *For any real number* α, *if* $\neg(\alpha \neq 0)$, *then* $\alpha = 0$.

[2] Kelly's proof may be found in [Cox48] or [Cox61, pp. 65-66].

[3] Constructive methods are described fully in [B67, B73, BB85]; see also [R82, BriMin84, M85, R99].

[4] For more details, and other constructive properties of the real number system, see [B67, BB85, BV06].

(b) *For any real number* α, *if* $\neg(\alpha > 0)$, *then* $\alpha \leq 0$; *if* $\neg(\alpha < 0)$, *then* $\alpha \geq 0$.

(c) *Let* α and β *be any real numbers, with* $\alpha < \beta$. *For any real number* x, *either* $x > \alpha$ *or* $x < \beta$.

(d) *Let* α and β *be any real numbers, with* $\alpha \neq \beta$. *For any real number* x, *either* $x \neq \alpha$ *or* $x \neq \beta$.

Property (c), known as the *Constructive Dichotomy Principle,* serves as a constructive substitute for the classical *Trichotomy Property*, which is constructively invalid. Property (d), which follows from (c), is called *cotransitivity;* it is the classical contrapositive of the transitive relation for equality.

Points A and B on the real plane $\mathbb{R}^2$ are *distinct*, written $A \neq B$, if the distance between them is positive. Points inherit properties from their coördinates; thus we have cotransitivity for points:

(e) *Let* A *and* B *be any points on the real plane* $\mathbb{R}^2$, *with* $A \neq B$. *For any point* X, *either* $X \neq A$ *or* $X \neq B$.

The condition P *lies on* l, written $P \in l$, means that the distance $\rho(P, l)$, from the point P to the line l, is 0, while P *lies outside* l, written $P \notin l$, means that the distance is positive.

The maximum and minimum of two real numbers are easily defined, using Cauchy sequences. For any real number α, we define $\alpha^+ = \max\{\alpha, 0\}$ and $\alpha^- = \max\{-\alpha, 0\}$. Attempting to construct the maximum or minimum of an arbitrary set of real numbers may, however, lead to interesting complications. We say that a set is *finite* if its elements may be listed, $\{a_1, a_2, ..., a_n\}$, with $n \geq 1$, but not necessarily distinctly.[5] A finite set of real numbers has a minimum, but an arbitrary nonvoid subset of a finite set of real numbers need not have a constructive minimum. Even when a minimum does exist, it need not be attained by an element in the set. These nonconstructivities will be established in Section 5.

3. Constructively invalid statements

Brouwerian counterexamples display the nonconstructivities in a classical theory, indicating feasible directions for constructive work. To illustrate the method, we give first an informal example on the plane $\mathbb{R}^2$.

Example. If, for the real plane $\mathbb{R}^2$, there is a proof of the statement

For any point P *and any line* l, *either* P *lies on* l, *or* P *lies outside* l,

then we have a method that will either prove the *Goldbach Conjecture,* or construct a counterexample.

Proof. Using a simple finite routine, construct a sequence $\{a_n\}_{n \geq 2}$ such that $a_n = 0$ if $2n$ is the sum of two primes, and $a_n = 1$ if it is not. Now apply the statement in question to the point $P = (0, \Sigma a_n/n^2)$, with the x-axis as the line l. If P lies on l, then we have proved the Goldbach Conjecture, while if P lies outside l, then we have constructed a counterexample. $\square$

[5] Several different definitions and variations of the concept *finite* may be found in the constructive literature. The definition here is close to the usual meaning of the term.

For this reason, such statements are said to be *constructively invalid.* If the Goldbach question is settled someday, then other famous problems may still be "solved" in this way. The example above will be applied in Section 5, locating one of the substantial nonconstructivities in a classical proof of the Sylvester-Gallai Theorem.

A *Brouwerian counterexample* is a proof that a given statement implies an omniscience principle. In turn, an *omniscience principle* would imply solutions or significant information for a large number of well-known unsolved problems. This method was introduced by Brouwer [Bro08] to demonstrate that reliance on the *Law of Excluded Middle* inhibits mathematics from attaining its full significance. The omniscience principles are special cases of LEM.

Omniscience principles are formulated in terms of binary sequences. The zeros and ones may represent the results of a search for a solution to a specific problem, as in the example above. These principles also have equivalent statements in terms of real numbers. The following omniscience principles will be used here:

Limited principle of omniscience (LPO). *For any binary sequence* $\{a_n\}$, *either* $a_n = 0$ *for all* n, *or there exists an integer* n such that $a_n = 1$. Equivalently, *For any real number* α *with* $\alpha \geq 0$, *either* $\alpha = 0$ *or* $\alpha > 0$.

Lesser limited principle of omniscience (LLPO). *For any binary sequence* $\{a_n\}$, *either the first integer* n *such that* $a_n = 1$ *(if one exists) is even, or it is odd.* Equivalently, *For any real number* α, *either* $\alpha \leq 0$ *or* $\alpha \geq 0$.

A statement is considered *constructively invalid* if it implies an omniscience principle. The example above, with slight modification, shows that the statement in question implies LPO; thus the statement is constructively invalid. When omniscience principles are involved, there is no need to mention a specific unsolved problem. LPO would solve a great number of unsolved problems; LLPO would provide unlikely partial solutions.[6]

4. Brouwerian counterexample

This will demonstrate that the Sylvester-Gallai Theorem is constructively invalid for the plane $\mathbb{R}^2$.

Example. The statement,

> *For any finite, noncollinear set of points* $\mathscr{S}$ *on the real plane* $\mathbb{R}^2$, *there exists a line that passes through exactly two points of* $\mathscr{S}$,

is constructively invalid; the statement implies LLPO.

[6] For more details concerning Brouwerian counterexamples, and other omniscience principles, see [B73, R02, M89, BV06].

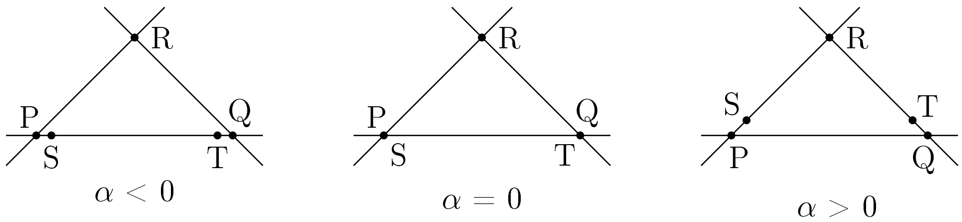

Figure 1. An omniscient view of the Brouwerian counterexample. Constructively, we do not know whether the real number α is < 0, or $= 0$, or > 0; the classical *Trichotomy Property* is constructively invalid. Since we do not know about α, we cannot specify a line that passes through exactly two points.

Proof. Let α be any real number; either $|\alpha| > 0$ or $|\alpha| < 10^{-20}$. Under the first condition, either $\alpha < 0$ or $\alpha > 0$, so we already have the required conclusion; thus we may assume the second condition. (Although the condition $|\alpha| < 1/2$ would suffice to keep the points in view, the condition chosen may bring to mind some unsolved problem in number theory that has been checked a long way out.)

Define $\mathscr{S} = \{P, Q, R, S, T\}$, where $P = (-1, 0)$, $Q = (1, 0)$, $R = (0, 1)$, $S = (-1 + |\alpha|, \alpha^+)$, $T = (1 - |\alpha|, \alpha^+)$. By hypothesis, there exists a line l as specified in the statement; there are at most ten possibilities for l.

Consider first the case in which l is one of the four lines RP, RQ, RS, RT, and suppose that $\alpha > 0$. Then $\alpha^+ = |\alpha| > 0$, so the point S lies on the line RP, distinct from both points R and P. Similarly, the point T lies on the line RQ. Thus l contains three distinct points of $\mathscr{S}$, a contradiction. It follows that $\alpha \leq 0$.

Now consider the case in which l is one of the six lines PQ, PS, PT, QS, QT, ST, and suppose that $\alpha < 0$. Then $|\alpha| > 0$, while $\alpha^+ = 0$, so the points S and T lie on the line PQ, distinct from each other and from P and Q. Thus l contains four distinct points of $\mathscr{S}$, a contradiction. It follows that $\alpha \geq 0$.

Thus the statement implies LLPO. □

5. Classical proof

To obtain a constructive version of the Sylvester-Gallai Theorem, we first subject Kelly's classical proof to examination from a constructive viewpoint, locating the constructively invalid steps. Kelly's 1948 proof for the plane $\mathbb{R}^2$ is repeated below in brief form (it appears in [Cox48] and [Cox61, pp. 65-66]). We follow Bishop's suggestion, that theorems and proofs dependent on the *Law of Excluded Middle* be labeled as such.

Proof. [LEM] (L. M. Kelly) Let $\mathscr{S}$ be a finite, noncollinear set of points on the real plane $\mathbb{R}^2$, consider all the lines that join two distinct points of $\mathscr{S}$, and consider the distances from each point of $\mathscr{S}$ to each of these lines. The set of all those distances that are positive has a minimum d, attained by at least one line l and one point P. Let E denote the foot of the perpendicular dropped from the point P onto the line l; the segment PE then has length d. If there are three distinct points of $\mathscr{S}$ that lie on l, then two of these must lie on the same side of E. Denote these two points by U and V, with U closer to E (possibly at E). Let h be the distance from the point U to the line PV; then h is one of the positive distances considered in determining the minimum d. However, it is apparent that h is less than d, a contradiction. Thus l contains exactly two points of $\mathscr{S}$. $\square$

We encounter several constructive obstacles in this remarkable proof. First, there is a problem in joining points to form the connecting lines, since we cannot determine which pairs of given points are distinct; considering distances, this would directly involve LPO.

If we manage to clear the first obstacle, there is then the problem of determining which points of $\mathscr{S}$ are at a positive distance from which of the connecting lines; such a determination was shown to be constructively invalid by the example in Section 3. Thus we are unable to list the positive distances as a finite set of real numbers, in order to construct a minimum.

Although the set of positive distances may be a nonvoid subset of a finite set of real numbers, it would be constructively invalid to claim that such a set has a minimum. For a Brouwerian counterexample, assume that such minimums exist, let α be a real number with $\alpha \geq 0$, and define $S = \{\alpha, 1\}$. Either $\alpha > 0$ or $\alpha < 1/2$; it suffices to assume the latter condition. Define $T = \{x \in S : x > 0\}$; then T is a nonvoid subset of the finite set S, so by hypothesis we may define $m = \min T$. Either $m > 1/2$ or $m < 1$. In the first case, $\neg(\alpha > 0)$, so $\alpha = 0$. In the second case, since m may be closely approximated by elements of T, it follows that $\alpha \in T$, so $\alpha > 0$. Thus LPO results. In the proof above, if we cannot show that the set of positive distances is finite, then we cannot construct a minimum.

Assuming the resolution of the above difficulties, there remains a problem with the selection of the special point-line pair. Even when the minimum of a set of real numbers can be constructed, still we cannot say that this minimum is attained by an element of the set. For a Brouwerian counterexample, assume that minimums are attained, let α be a real number, define $S = \{\alpha^+, \alpha^-\}$, and define $m = \min S$. By hypothesis, either $m = \alpha^+$, and then $\neg(\alpha > 0)$, so $\alpha \leq 0$, or $m = \alpha^-$, and then $\neg(\alpha < 0)$, so $\alpha \geq 0$. Thus LLPO results. Thus a minimum is, in general, not attained; we may only construct elements of the set arbitrarily close to the minimum. In the proof above, selecting the point P and the line l, for only an approximation e to the minimum d, would disturb the measurements, since then h, while less than e, would not be known to be less than d.

The final constructive obstacle concerns the situation in which three points of the set $\mathscr{S}$ are assumed to lie on the selected line l; the proof must decide on which side of the foot E each point lies. This amounts to deciding on which side of zero an arbitrary real number lies, and this is precisely the nonconstructive omniscience principle LLPO.

In adapting Kelly's proof, these constructive obstacles will be overcome by adding an hypothesis, by selecting a sufficiently close approximation to the minimum distance, and by using constructive properties of the real numbers.

6. Constructive version

Since the Sylvester-Gallai Theorem is constructively invalid, a constructive version must be restricted to a set of points with additional structure. A set is *discrete* if any two elements are either equal or distinct. A set of points $\mathscr{S}$ is *linearly discrete* if for any point P in $\mathscr{S}$, and for any line l connecting two distinct points of $\mathscr{S}$, either P lies on l, or P lies outside l.

For any set of points, these two conditions follow from LEM; thus the constructive version of the theorem will be classically equivalent to the traditional version. Although we require both additional conditions, it will suffice to specify only the second, as the lemma below will demonstrate.

A strong definition will be used for a *noncollinear* set of points: the set contains distinct points P, Q, R such that P lies outside the line QR. The definition of *finite* set will be as given in Section 2.

Lemma. *If a noncollinear set $\mathscr{S}$ of points on the real plane $\mathbb{R}^2$ is linearly discrete, then it is discrete.*

Proof. Let A and B be any points of $\mathscr{S}$.

Select three distinct, noncollinear points C_1, C_2, C_3 of $\mathscr{S}$. By cotransitivity, one of these, call it P, is distinct from A. The point B either lies outside the line AP or it lies on AP. In the first case, we have $B \neq A$, so there remains only the case in which $B \in AP$.

Each of the points C_i either lies on the line AP or lies outside AP. These noncollinear points cannot all lie on AP, so there exists one, call it Q, with $Q \notin AP$; thus $Q \neq B$. The point A either lies outside the line BQ or it lies on BQ. In the first case, we have $A \neq B$ again, so only the case in which $A \in BQ$ remains.

Since $Q \notin AP$, the lines AP and BQ are distinct. The points A and B are both common to each of these lines; hence $A = B$.

The conclusion in the following constructive version of the Sylvester-Gallai Theorem is stronger than that usually seen in classical versions; it gives the result in an affirmative form. Rather than merely showing that it is impossible for a third point of the given set to lie on the selected line, the proof shows that any point of the set that lies on the selected line must be identical with either one or the other of the two selected points. □

Theorem. *Let $\mathscr{S}$ be any finite, linearly discrete, noncollinear set of points on the real plane $\mathbb{R}^2$. There exist distinct points A, B in $\mathscr{S}$ such that the line $l = AB$ passes through only these two points of $\mathscr{S}$; for any point X of $\mathscr{S}$ that lies on l, either $X = A$ or $X = B$.*

Proof. (i) The lemma shows that the family $\mathscr{S}$ is discrete; thus the family $\mathscr{P}$, of all pairs (Q, R) of distinct points in $\mathscr{S}$, is finite. The condition *linearly discrete* then ensures that, for each pair (Q, R) in $\mathscr{P}$, the set of points P in $\mathscr{S}$, with P lying outside the line QR, is also finite. Thus, from the family of all ordered triads (P, Q, R) of points of $\mathscr{S}$, we may select and list those triads with both properties, that Q and R are distinct, and that P lies outside the line QR. Hence the set of triads

$$\mathscr{T} = \{(P, Q, R) : P, Q, R \in \mathscr{S},\ Q \neq R,\ P \notin QR\}$$

is finite, and the set

$$\mathscr{D} = \{\rho(P, QR) : (P, Q, R) \in \mathscr{T}\}$$

of positive distances is also finite.

Define $d = \min \mathscr{D}$, and let D be the diameter of $\mathscr{S}$. Since $\mathscr{D}$ is finite, the minimum exists constructively, but the counterexample in Section 5 shows that this minimum need not be attained by an element of $\mathscr{D}$, although close approximations may be found. Select a real number e in $\mathscr{D}$, where $e = \rho(K, AB)$ and $(K, A, B) \in \mathscr{T}$, such that $e < d\sqrt{1 + d^2/D^2}$. Define $l = AB$, and let F denote the foot of the perpendicular dropped from the point K onto the line l.

(ii) Coördinatize the line l so that the point F has coördinate 0, and the metric ρ of $\mathbb{R}^2$ is preserved. Let the coördinates of the points A and B be denoted a and b. Since $A \neq B$, we have $a \neq b$. By cotransitivity, either $a \neq 0$ or $b \neq 0$. By symmetry, it suffices to consider the first case. Reversing the coördinatization if needed, we may assume that $a > 0$.

(iii) *If the points Y and Z of $\mathscr{S}$ lie on the line l with coördinates y and z such that $y \geq 0$ and $z \geq 0$, then $Y = Z$.*

To prove this, suppose that $y \neq z$; it suffices to consider the case in which $z < y$. Denote by G the foot of the perpendicular dropped from the point Z onto the line YK, and define $h = \rho(Z, G)$. Considering the similar right triangles YZG and YKF, we have

$$h/y \leq h/(y - z) = e/\sqrt{y^2 + e^2}$$

(The inequality relating the extreme terms may also be obtained by comparing the areas of the triangles YZK and YFK.) Since $d \leq e$ and $y \leq \rho(Y, K) \leq D$, it follows that

$$h \leq e/\sqrt{1 + e^2/y^2} \leq e/\sqrt{1 + d^2/D^2} < d$$

However, since (Z, Y, K) belongs to the set of triads $\mathscr{T}$, we have $h \in \mathscr{D}$, so $d \leq h$, a contradiction. Thus $y = z$.

(iv) Suppose that $b > 0$. By (iii), we then have $B = A$, a contradiction. Thus $b \leq 0$.

(v) Now let X be any point of $\mathscr{S}$ that lies on l, with coördinate x. By cotransitivity, either $X \neq A$ or $X \neq B$. In the first case, suppose that $x > 0$. Then it follows from (iii) that $X = A$, a contradiction. This shows that $x \leq 0$, and thus, by (iii)(reversed), we have $X = B$. Similarly, in the second case we find that $X = A$. □

There are other classical versions of the Sylvester-Gallai Theorem, and other proofs, which might be constructivized; see [KelMos58, Wil68, Cha70, Lin88, BorMos90, Chv04, Pam09].

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References

[B67] E. Bishop, *Foundations of Constructive Analysis,* McGraw-Hill, New York, 1967. MR0221878

[B73] E. Bishop, *Schizophrenia in Contemporary Mathematics*, AMS Colloquium Lectures, Missoula, Montana, 1973. Reprinted in *Contemporary Mathematics* 39:l-32, 1985. MR0788163

[BB85] E. Bishop and D. Bridges, *Constructive Analysis,* Springer-Verlag, Berlin, 1985. MR0804042

[BR87] D. Bridges and F. Richman, *Varieties of Constructive Mathematics,* Cambridge University Press, Cambridge, UK, 1987. MR0890955

[BV06] D. Bridges and L. Vîţă, *Techniques of Constructive Analysis,* Springer, New York, 2006. MR2253074

[BorMos90] P. Borwein and W. O. J. Moser, A survey of Sylvester's problem and its generalizations, *Aequationes Math.* 40:111–135, 1990. MR1069788

[BriMin84] D. Bridges and R. Mines, What is constructive mathematics?, *Math. Intelligencer* 6:32–38, 1984. MR0762057

[Bro08] L. E. J. Brouwer, De onbetrouwbaarheid der logische principes, *Tijdschrift voor Wijsbegeerte* 2:152-158, 1908. English translation, "The Unreliability of the Logical Principles", pp. 107–111 in A. Heyting (ed.), *L. E. J. Brouwer: Collected Works 1: Philosophy and Foundations of Mathematics,* Elsevier, Amsterdam-New York, 1975. MR0532661

[Cha70] G. D. Chakerian, Sylvester's problem on collinear points and a relative, *Amer. Math. Monthly* 77:164–167, 1970. MR0258659

[Chv04] V. Chvátal, Sylvester-Gallai theorem and metric betweenness, *Discrete Comput. Geom.* 31:175–195, 2004. MR2060634

[Cox48] H. S. M. Coxeter, A problem of collinear points, *Amer. Math. Monthly* 55:26–28, 1948. MR0024137

[Cox61] H. S. M. Coxeter, *Introduction to Geometry,* Wiley, New York, 1961. MR0123930

[Erd43] P. Erdős, Problem for solution 4065, *Amer. Math. Monthly* 50:65, 1943.

[Erd82] P. Erdős, Personal reminiscences and remarks on the mathematical work of Tibor Gallai, *Combinatorica* 2:207–212, 1982. MR0698647

[KelMos58] L. M. Kelly and W. O. J. Moser, On the number of ordinary lines determined by n points, *Canad. J. Math.* 10:210–219, 1958. MR0097014

[Lin88] X. B. Lin, Another brief proof of the Sylvester theorem, *Amer. Math. Monthly* 95:932–933, 1988. MR979138

[M85] M. Mandelkern, Constructive mathematics, *Math. Mag.* 58:272-280, 1985. MR0810148

[M89] M. Mandelkern, Brouwerian counterexamples, *Math. Mag.* 62:3-27, 1989. MR0986618

[Mel41] E. Melchior, Über Vielseite der projektiven Ebene, *Deutsche Math.* 5:461-475, 1941. MR0004476

[Pam09] V. Pambuccian, A reverse analysis of the Sylvester-Gallai theorem, *Notre Dame J. Form. Log.* 50:245–260, 2009. MR2572973

[Pla05] J. von Plato, A constructive approach to Sylvester's conjecture, *J.UCS* 11:2165–2178, 2005. MR2210695

[R82] F. Richman, Meaning and information in constructive mathematics, *Amer. Math. Monthly* 89:385-388, 1982. MR0660918

[R99] F. Richman, Existence proofs, *Amer. Math. Monthly* 106:303-308, 1999. MR1682389

[R02] F. Richman, Omniscience principles and functions of bounded variation, *MLQ Math. Log. Q.* 42:111-116, 2002. MR1874208

[Ste44] R. Steinberg, Three point collinearity, *Amer. Math. Monthly* 51:169-171, 1944.

[Syl93] J. J. Sylvester, Mathematical question 11851, *Educational Times* 59:98, 1893.

[Wil68] V. C. Williams, A proof of Sylvester's theorem on collinear points, *Amer. Math. Monthly* 75:980–982, 1968. MR1535105

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