SUPREMA OF LOCATED SETS

MARK MANDELKERN

1. *Introduction*

It is *not* always possible to measure the distance $\rho(x, G)$ between a point x and a set G in a metric space; sets for which the distance exists for all x are called *located*. The concept was introduced by Brouwer [**2**] and is used extensively in constructive analysis. For example, although *not* every nonvoid bounded set of real numbers has a supremum, each nonvoid, bounded, located set on the line is easily seen to have a supremum.

In the case of unbounded sets, suprema are sought in the system R^∞ of extended real numbers constructed in [**3**]. In sharp contrast to the bounded case, *not* all located sets have suprema in R^∞; an example was given in [**3**], where it was also shown that nonvoid, *convex* located sets do have suprema.

Here those located sets on the line which have suprema in R^∞ are characterized in several ways. The main characterization, Theorem 1 below, utilizes the characterization of located sets given in [**4**], where closed located sets G on the line are shown to be precisely those sets of the form

$$G = \bigcap_n((-\infty, a_n] \cup [b_n, +\infty))$$

where

(1) each open interval (a_n, b_n) is either void or nonvoid;

(2) (a_n, b_n) and (a_m, b_m) are disjoint whenever $n \neq m$;

(3) there exists a sequence $\{M_k\}$ of positive integers such that $n \leqslant M_k$ whenever (a_n, b_n) meets $(-k, k)$ and $b_n - a_n > 1/k$.

This is called the *notched representation* of the located set G, the intervals (a_n, b_n) are the *notches of* G, and the integers M_k are called the *locating parameters for* G.

2. *Existence of suprema*

It suffices to consider closed located sets. Recall that the notch (a_n, b_n) is represented by the extended real numbers a_n and b_n, for which, in general, it is not known whether they are finite or infinite. The following characterization of located sets with suprema shows that it is just this lack of information which prevents the construction of suprema for certain located sets. We follow the convention that the void interval is written as $(1, 0)$; this simplifies the statements of results, since then only the nonvoid notches can have possibly infinite endpoints.

THEOREM 1. *Let G be a closed located set on the line, and*

$$G = \bigcap_n((-\infty, a_n] \cup [b_n, +\infty))$$

Received 25 September, 1978; revised 4 January, 1979.

[J. LONDON MATH. SOC. (2), 20 (1979), 161–164]

its notched representation. Then G has a supremum in R^∞ *if and only if each* b_n *is either finite or infinite.*

Proof. First let $s \equiv \sup G$ exist. Considering any n, we may assume that $a_n < b_n$. If $s < b_n$, suppose that $b_n < +\infty$. Then b_n is a finite endpoint and $b_n \in G$, a contradiction. Hence $b_n = +\infty$. If $s > a_n$, then construct a point y in G such that $y > a_n$; thus $y \geqslant b_n$ and $b_n < +\infty$.

Now let $b_n = +\infty$ or $b_n < +\infty$, for all n. We may assume that the sequence $\{M_k\}$ of locating parameters is increasing. Two cases obtain, depending on whether or not $b_i = +\infty$ for some $i \leqslant M_1$. (This reflects the fact that when $s \equiv \sup G$ exists, either G is bounded above ($s < +\infty$) or G is nonvoid ($s > -\infty$).)

Case I. There exists $i \leqslant M_1$ with $b_i = +\infty$. In this case we show that $a_i = \sup G$. (a) Let $x \in G$. Since $x \geqslant b_i$ is impossible, it follows that $x \leqslant a_i$. (b) Let $t < a_i$. Then a_i is a finite endpoint, so $a_i \in G$.

Case II. $b_n < +\infty$ for all $n \leqslant M_1$. Define an increasing sequence $\{\sigma_k\}$ of zeros and ones as follows:

(i) put $\sigma_k \equiv 0$ if $b_n < +\infty$ for all $n \leqslant M_k$;

(ii) put $\sigma_k \equiv 1$ if $b_i = +\infty$ for some $i \leqslant M_k$.

Put $s_k \equiv k$ if $\sigma_k = 0$, and $s_k \equiv a_i$ when $\sigma_k = 1$. (Note that there can be at most one $b_i = +\infty$.)

We first use the sequence $\{s_k\}$ to define an extended real number s. In the case that $\sigma_k = 1$, suppose that $a_i < 1$. Then (a_i, b_i) meets $(-1, 1)$, so $i \leqslant M_1$, which is a contradiction and hence $a_i > -\infty$. Thus $\{s_k\}$ is a sequence of finite numbers. To show that $\{s_k\}$ is an extended Cauchy sequence, we apply [3; Theorem 3].

(1) Let $\sigma_k = 0$ and $n \geqslant k$. When $\sigma_n = 0$, then $s_n = n \geqslant k$. When $\sigma_n = 1$, then $s_n = a_i$. Suppose that $a_i < k$. Then (a_i, b_i) meets $(-k, k)$, so $i \leqslant M_k$ and $\sigma_k = 1$, which is a contradiction and hence $a_i \geqslant k$.

(2) Let $\sigma_k = 1$ and $n \geqslant k$. Then $s_n = s_k = a_i$.

Thus $\{s_k\}$ is an extended Cauchy sequence and $s \equiv \{s_k\}$ is an extended real number.

We now show that $s = \sup G$. (a) Let $x \in G$ and suppose that $x > s$. Then $s < +\infty$ so that there exists k such that $\sigma_k = 1$ and $s = a_i$. Thus $x \in (a_i, b_i)$, which is a contradiction and hence $x \leqslant s$. (b) Let $t < s$. Choose k so that $s_k - 1/k > t$. When $\sigma_k = 0$, consider first the case $\rho(k, G) > 1/2k$. Choose n so that $a_n < k < b_n$. Then $b_n - a_n > 1/k$, so $n \leqslant M_k$ and $b_n < +\infty$; thus $b_n \in G$ with $b_n > k = s_k > t$. Now consider the case $\rho(k, G) < 1/k$. Construct $y \in G$ so that $\rho(k, y) < 1/k$. Then $y > k - 1/k = s_k - 1/k > t$. When $\sigma_k = 1$, then $s = a_i$ and hence s lies in G.

The following result shows a relation between the located set G and the set $\bar{G}$, called the *compression of* G, that is, the image of G under the mapping $x \to \bar{x}$ which maps the line onto the interval $(-1, 1)$ [3; §3]. The proof is routine.

THEOREM 2. *A nonvoid located set G on the line has a supremum in* R^∞ *if and only if its compression* $\bar{G}$ *is located in* $(-1, 1]$.

3. *Nonvoid located sets*

Notched representations also provide a characterization of nonvoid located sets. The characterization below shows that the method of notches in [4] for constructing a located set, yields a nonvoid set provided that no notch is the entire line. That this intuitively obvious result applies only to located sets is shown by the example following the proof.

THEOREM 3. *Let G be a closed located set on the line with notches (a_n, b_n). Then G is nonvoid if and only if for every n, either a_n or b_n is finite.*

Proof. Let G be nonvoid and construct z in G. For any n, either $z \leqslant a_n$ or $z \geqslant b_n$; thus either a_n or b_n is finite.

Now let a_n or b_n be finite for every n. If $\rho(0, G) > 0$ then some notch is nonvoid and thus has a finite endpoint, which lies in G. If $\rho(0, G) < +\infty$, then again G is nonvoid.

The following example shows that the above result does not extend to an arbitrary set of the form given by a notched representation, even when conditions (1) and (2) in §1 are satisfied. Thus the locating parameters of condition (3) are essential. Let $\{\alpha_n\}$ be an increasing sequence of zeros and ones. Put $(a_n, b_n) \equiv (1, 0)$ unless n is the first integer with $\alpha_n = 1$; in that case put $(a_n, b_n) \equiv (-n, n)$. Although the points a_n and b_n satisfy the condition of the theorem, the set $G = \bigcap_n((-\infty, a_n] \cup [b_n, +\infty))$ is *not* nonvoid.

4. *Convex sets*

An application of Theorem 1 yields a simple proof of [3; Theorem 4].

COROLLARY. *Every nonvoid convex located set on the line has a supremum and an infimum.*

Proof. We may assume that G is closed, since it is easily seen that the closure of any convex set is also convex. Construct a point z in G and let $a_n < b_n$. If $z \leqslant a_n$, then $a_n \in G$. Suppose that $b_n < +\infty$; then $b_n \in G$, contradicting the convexity of G. Hence $b_n = +\infty$. If $z \geqslant b_n$, then $b_n < +\infty$.

The condition that G be nonvoid can *not* be dropped; let $\{\alpha_n\}$ be an increasing sequence of zeros and ones and consider the set G consisting of all j such that $\alpha_j < \alpha_{j+1}$. Since G contains at most one point, it is convex. It is also located; still it has *no* supremum in R'. This is another aspect of the fact, noted in the proof of Theorem 1, that a set with a supremum is either nonvoid or bounded above.

5. *Reciprocal sets*

Here we relate the existence of the supremum of a located set G of positive points on the line with the *reciprocal set* $1/G$; first we must consider reciprocals of extended real numbers.

THEOREM 4. *For all extended real numbers t with $0 \leqslant t \leqslant +\infty$ there correspond extended real numbers $\hat{t}$, also with $0 \leqslant \hat{t} \leqslant +\infty$, such that*

(i) $t < u$ *if and only if* $\hat{u} < \hat{t}$;

(ii) $t \leqslant u$ *if and only if* $\hat{u} \leqslant \hat{t}$;

(iii) $\hat{\hat{t}} = t$;

(iv) $\hat{0} = +\infty$ *and* $\widehat{+\infty} = 0$;

(v) *if* $0 < t < +\infty$, *then* $\hat{t} = 1/t$.

Proof. For any t, choose an extended Cauchy sequence $\{t_n\}$ of positive numbers with $t = \{t_n\}$. Then $\{1/t_n\}$ is also an extended Cauchy sequence; put $\hat{t} \equiv \{1/t_n\}$ and the results follow.

Definition. The extended real number $\hat{t}$ will be called the *extended reciprocal of t*, and will be denoted by $1/t$.

The set H is said to *approximate the set G to within* ε if for every point x of G (respectively H) there is a point y of H (respectively G) such that $\rho(x, y) < \varepsilon$. The proof of the following lemma is routine.

LEMMA. *Let G be a set in a metric space X. If for every* $\varepsilon > 0$ *there is a located set H in X approximating G to within* ε, *then G is located.*

THEOREM 5. *Let G be a located set of positive points on the line. Then G has a supremum if and only if* $1/G$ *is located.*

Proof. First let $t \equiv \sup G$ exist. If $t < 0$, then G is void. If $t > -\infty$, then G is nonvoid; hence we may apply the characterization of nonvoid located sets given in [**3**; Theorem 1]. It suffices to construct, for any positive integer n, a number $b > n$ such that $G_1 \equiv (1/G) \cap [0, b]$ is located. Considering $\rho(0, G)$, construct $b > n$ such that $1/b \in G$ or $1/b \in -G$. It suffices to construct, for any $\varepsilon > 0$, a located set H approximating G_1 to within ε; we may assume that $\varepsilon < b$. When $t < +\infty$, G is bounded, hence $H \equiv G_1$ is located by the continuity of the mapping $x \to 1/x$ and [**3**; Lemma 7]. When $t > 1/\varepsilon$, construct $y \in G$ so that $y > 1/\varepsilon$. Then $1/b < y$, so that $G \cap [1/b, y]$ is located. Thus $H \equiv (1/G) \cap [1/y, b]$ is located and approximates G_1 to within ε.

Now let $1/G$ be located. Put $t \equiv \rho(0, 1/G)$. When $t > 0$, G is bounded; thus some $b_i = +\infty$. Thus $a_i = \sup G$. When $t < +\infty$, then $1/t = \sup G$: (a) Let $x \in G$. Then $1/x \in 1/G$, so $t \leqslant 1/x$, and $x \leqslant 1/t$. (b) Let $s < 1/t$. Since $1/t > 0$, we may assume that $s \geqslant 0$. Then $t < 1/s$. Construct $y \in G$ such that $1/y < 1/s$. Hence $y > s$.

References

1. E. Bishop, *Foundations of constructive analysis*, McGraw-Hill, New York, 1967.
2. L. E. J. Brouwer, Begründung der Mengenlehre Unabhängig vom Logischen Satz vom Ausgeschlossenen Dritte II: Theorie der Punktmengen, Ver. Kon. Akad. Wet. (Amsterdam) 12 (1919), *nr.* 7.
3. M. Mandelkern, Resolutions on the line, *Pacific J. Math.*, to appear.
4. M. Mandelkern, Located sets on the line, to appear.

Birkbeck College,
London.
and
New Mexico State University,
Las Cruces,
New Mexico 88003,
U.S.A.