

The Altitudes of a Triangle

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Abstract. A long-standing, unanswered question regarding Euclid's *Elements* concerns the absence of a theorem for the concurrence of the altitudes of a triangle, and the possible reasons for this omission; this suggests a search for the most elementary and direct proof. This paper provides a simple, direct, elementary proof of the theorem, based solely on the *Elements*.

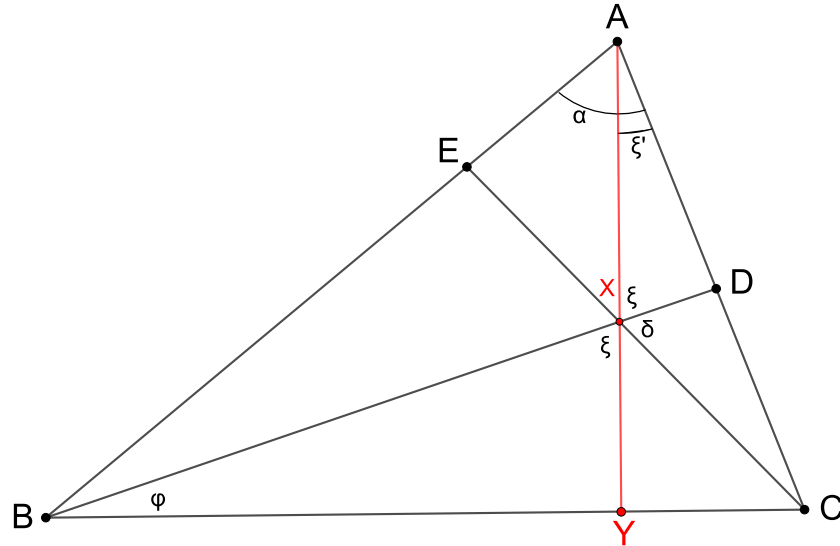
The concurrence of the angle bisectors of a triangle, at the *incenter*, follows directly from Proposition 4 in Book IV of Euclid's *Elements* [2], and the concurrence of the perpendicular bisectors of the sides, at the *circumcenter*, follows directly from Proposition 5. In the extant Euclid, however, there is no mention anywhere of the concurrence of the altitudes, at the *orthocenter*. This omission has long been a source of mystery and speculation. In the centuries following Euclid, a remarkable number of proofs have been put forward; this suggests a search for the most elementary and direct proof.

A history of the altitude theorem, a comprehensive discussion of known proofs, and an extensive list of references will be found in [1]. Several proofs listed there involve the drawing of additional lines, some depend on a concurrence theorem for the incenter or circumcenter, and others depend on Ceva's Theorem, cyclic quadrilaterals, vectors, norms, complex numbers, trigonometry, analytic geometry, Cartesian coordinates, and other methods introduced only in recent centuries. The most elementary of the listed proofs is Proof 2, based on the theory of similar triangles; it is attributed to Newton.

The proof here, while also involving similar triangles, differs from Newton's proof in a fundamental manner. The proof attributed to Newton notes the intersection points of two different pairs of altitudes, and shows that these points coincide. The proof here draws only two of the altitudes, and then proves that the line drawn from the third vertex, through the point where these two meet, is the third altitude; this technique is more direct and elementary.

The proof below is simple, elementary, and direct; it introduces no additional lines, depends on none of the advanced methods of modern times, and respects the axiomatic character of the *Elements*.

Note. This revision of the manuscript includes a minor update; eliminating one unnecessary step and one label in the drawing, the proof of the condition $\alpha = \delta$ is now more direct.



Theorem. *The altitudes of a triangle are concurrent.*

Proof. Let ABC be an acute-angled triangle. (The proof for an obtuse triangle will be essentially the same, *mutatis mutandis*. Alternatively, the Lemma in [1] could be used to reduce the problem to acute triangles.)

Draw the altitudes BD and CE , intersecting at the point X , and draw the line from the vertex A , through the point X , meeting side BC at the point Y . Write $\varphi = \angle CBD$, and $\xi = \angle BXY$; clearly, $\angle AXD = \xi$, and $\angle XAD = \xi'$, the complement of ξ . The angles $\alpha = \angle CAE$ and $\delta = \angle CXD$ are each complementary to angle $\angle ACE$; thus $\alpha = \delta$.

The right triangles ABD and XCD have the corresponding equal acute angles α, δ ; thus they are equal-angled. The proportion¹ $AD : BD :: XD : CD$ follows from [VI,4].² Noticing that the alternate proportion $AD : XD :: BD : CD$, derived from [V,16], expresses a proportionality for the sides of the right triangles AXD and BCD , and applying [VI,6], we find that these triangles are equal-angled; thus $\varphi = \xi'$.

Now two of the angles in triangle BXY are complementary; thus the third angle $\angle BYX$ is a right angle. This shows that the line AY is the third altitude. \square

References

- [1] Hajja, M., Martini, H., Concurrency of the altitudes of a triangle, *Math. Semesterber.* 60(2013), 249–260.
- [2] Heath, T. L., *The Thirteen Books of Euclid's Elements*, 2nd ed., 3 volumes, Cambridge University Press, 1926. Reprint, Dover, 1956. Reprint, Cambridge University Press, 2015.

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¹In keeping with the character of the *Elements*, this proportion is written in a manner which avoids reference to algebra in the modern sense.

²This notation refers to Book VI, Proposition 4, in Euclid's *Elements* [2].