Constructive Projective Geometry

Mark Mandelkern

Preprint 2019 December 27

To appear in *Handbook of Constructive Mathematics,* ed., D. Bridges, et al, Cambridge University Press, 2023, Chapter 7, pp. 168-198.

New Mexico State University
http://www.zianet.com/mandelkern
mandelkern@zianet.com

Contents

1 Introduction

Of the great theories of classical mathematics, projective geometry, with its powerful concepts of symmetry and duality, has been exceptional in continuing to intrigue investigators. The challenge put forth by Errett Bishop (1928-1983),

> Every theorem proved with [nonconstructive] methods presents a challenge: to find a constructive version, and to give it a constructive proof. [B67, p. x; BB85, p. 3]

and Bishop's "Constructivist Manifesto" [B67, Chapter 1; BB85, Chapter 1], motivate a large portion of current constructive work. This challenge can be answered by discovering the hidden constructive content of classical projective geometry. Here we briefly outline, with few details, recent constructive work on the real projective plane, and projective extensions of affine planes. Special note is taken of a number of interesting open problems that remain; these show that constructive projective geometry is still a theory very much in need of further effort.

There has been a considerable amount of work in the constructivization of geometry, on various topics, in different directions, and from diverse standpoints. For the constructive extension of an affine plane to a projective plane, see [H59, vDal63, M13, M14]. For the constructive coördinatization of a plane, see [M07]. Intuitionistic axioms for projective geometry were introduced by A. Heyting [H28], with further work by D. van Dalen [vDal96]. Work in constructive geometry by M. Beeson [Be10, Be16] uses Markov's Principle, a nonconstructive principle which is accepted in recursive function theory, but not in the Bishop-type strict constructivism that is adhered to in the present paper. M. Lombard and R. Vesley [LV98] construct an axiom system for intuitionistic plane geometry, and study it with the aid of recursive function theory. The work of J. von Plato [vPla95, vPla98, vPla10] in constructive geometry, proceeding from the viewpoint of formal logic, is related to type theory, computer implementation, and combinatorial analysis. The work of V. Pambuccian, e.g., [Pam98, Pam01, Pam03, Pam05, Pam11], also proceeding within formal logic, covers a wide range of topics concerning axioms for constructive geometry.

The Bishop-type constructive mathematics discussed in the present paper proceeds from a viewpoint well-nigh opposite that of either formal logic or recursive function theory. For further details concerning this distinction, see [B65, B67, B73, B75, BB85, BR87].

Part I
Real projective plane

Arend Heyting (1898-1980), in his doctoral dissertation [H28], began the constructivization of projective geometry. Heyting's work involves both synthetic and analytic theories. Axioms for projective space are adopted; since a plane is then embedded in a space of higher dimension, it is possible to include a proof of Desargues's Theorem. For the coördinatization of projective space, axioms of order and continuity are assumed. The theory of linear equations is included, and results in analytic geometry are obtained. Later, Heyting discussed the role of axiomatics in constructive mathematics as follows:

> At first sight it may appear that the axiomatic method cannot be used in intuitionistic mathematics, because there are only considered mathematical objects which have been constructed, so that it makes no sense to derive consequences from hypotheses which are not yet realized. Yet the inspection of the methods which are actually used in intuitionistic mathematics shows us that they are for an important part axiomatic in nature, though the significance of the axiomatic method is perhaps somewhat different from that which it has in classical mathematics. [H59, p. 160]

Recent work [M16, M18], briefly outlined below, constructivized the synthetic theory of the real projective plane as far as harmonic conjugates, projectivities, the axis of homology, conics, Pascal's Theorem, and polarity. Axioms only for a plane are used. The basis for the constructivization is the extensive literature concerning the classical theory, including works of O. Veblen and J. W. Young [VY10, Y30], H. S. M. Coxeter [Cox55], D. N. Lehmer [Leh17], L. Cremona [Cre73], and G. Pickert [Pic75]. An entertaining history of the classical theory is found in Lehmer's last chapter.

2 Axioms

For nearly two hundred years a sporadic and sometimes bitter debate has continued, concerning the value of synthetic versus analytic methods. In his Erlangen program of 1872, Felix Klein sought to mediate the dispute:

> The distinction between modern synthetic and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form in both. We chose therefore as common designation of them both the term *projective geometry.*

> Although the synthetic method has more to do with space-perception and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulae of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations. On the other hand, the advantage to original research of a well formulated analysis should not be underestimated, - an advantage due to its moving, so to speak, in advance of the thought. But it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aid of analysis is only a first, though a very important, step. [Kle72]

In the synthetic work summarized below, axioms are formulated which can be traced to an analytic model based on constructive properties of the real numbers, and the resulting axiom system is used to construct a synthetic projective plane $\mathbb{P}$. In this sense, the construction of the plane $\mathbb{P}$ takes into account Bishop's thesis: "All mathematics should have numerical meaning" [B67, p. ix; BB85, p. 3].

2.1 Axiom Group C

The constructivization of [M16], resulting in the projective plane $\mathbb{P}$, uses only axioms for a plane. There exist non-Desarguesian projective planes; see, for example, [Wei07]. This means that Desargues's Theorem must be taken as an axiom; it is required to establish the essential properties of harmonic conjugates. Other special features of the axiom system are also required, to obtain constructive versions of the most important classical results. The consistency of the axiom system is verified by means of an analytic model, discussed below in Section 8; the properties of this model have guided the choice of axioms.

The constructive axiom group C, adopted for the projective plane $\mathbb{P}$ in [M16, Section 2], has seven initial axioms. The first four are those usually seen for a classical projective plane; e.g., two points determine a line, and two lines intersect at a point. The last three axioms, which have special constructive significance, will be discussed below.

For the construction of the projective plane $\mathbb{P}$, there is given a family $\mathscr{P}$ of points and a family $\mathscr{L}$ of lines, along with equality and inequality relations for each family. The *inequality relations* assumed for the families $\mathscr{P}$ and $\mathscr{L}$, both denoted $\neq$, are *tight apartness relations*; thus, for any elements x, y, z, the following conditions are satisfied:

(i) $\neg(x \neq x)$.
(ii) If $x \neq y$, then $y \neq x$.
(iii) If $x \neq y$, then either $z \neq x$ or $z \neq y$.
(iv) If $\neg(x \neq y)$, then $x = y$.

The notion of an apartness relation was introduced by Brouwer [Brou24], and developed further by Heyting [H66]. Property (iii) is known as *cotransitivity*, and (iv) as *tightness*. The implication "$\neg(x = y)$ implies $x \neq y$" is invalid in virtually all constructive theories, the inequality being the stronger of the two conditions. For example, with real numbers considered constructively, $x \neq 0$ means that there exists an integer n such that $1/n < |x|$, whereas $x = 0$ means merely that it is contradictory that such an integer exists. For more details concerning the constructive properties of the real numbers, see [B67, BB85, BV06]; for a comprehensive treatment of constructive inequality relations, see [BR87, Section 1.2].

A given *incidence relation,* written $P \in l$, links the two families; we say that *the point P lies on the line* l, or that l *passes through* P. A line is not viewed as a set of points; the set $\bar{l}$ of points that lie on a line l is a *range of points*, while the set Q^* of lines that pass through a point Q is a *pencil of lines.* The outside relation $P \notin l$ is obtained by a definition:

Definition. OUTSIDE RELATION. For any point P on the projective plane $\mathbb{P}$, and any line l, it is said that P *lies outside* l (and l *avoids* P), and written $P \notin l$, if $P \neq Q$ for all points Q that lie on l. [M16, Defn. 2.3]

This condition for the relation $P \notin l$, when viewed classically, is simply the negation of the condition $P \in l$, when written as the tautology "there exists $Q \in l$ such that $P = Q$". Constructively, however, the condition acquires a strong, positive significance, derived from the character of the condition $P \neq Q$.

Several axioms connect these relations:

Axiom C5. *For any lines l and m on the projective plane $\mathbb{P}$, if there exists a point P such that $P \in l$, and $P \notin m$, then $l \neq m$.*

The implication "If $\neg(P \in l)$, then $P \notin l$" is nonconstructive. However, we have:

Axiom C6. *For any point P on the projective plane $\mathbb{P}$, and any line l, if $\neg(P \notin l)$, then $P \in l$.*

Axiom C6 would be immediate in a classical setting, where $P \notin l$ means $\neg(P \in l)$; applying the law of excluded middle, a double negation results in an affirmative statement. For a constructive treatment, where the condition $P \notin l$ is not defined by negation, but rather by the affirmative definition above, Axiom C6 must be assumed; it is analogous to the tightness property of the inequality relations that are assumed for points and lines.

For the metric plane $\mathbb{R}^2$, the condition of Axiom C6 follows from the analogous constructive property of the real numbers: "For any real number α, if $\neg(\alpha \neq 0)$, then $\alpha = 0$", interpreting the outside relation in terms of distance. For the analytic model $\mathbb{P}^2(\mathbb{R})$, which motivates the axiom system, Axiom C6 is verified using this constructive property of the real numbers.

The following axiom has a preëminent standing in the axiom system; it is indispensable for virtually all constructive proofs involving the projective plane $\mathbb{P}$. The point of intersection of distinct lines l and m is denoted $l \cdot m$.

Axiom C7. *If l and m are distinct lines on the projective plane $\mathbb{P}$, and P is a point such that $P \neq l \cdot m$, then either $P \notin l$ or $P \notin m$.*

This axiom is a strongly worded, yet classically equivalent, constructive form of a classical axiom: "distinct lines have a *unique* common point", which means only that if the points P and Q both lie on both lines, then $P = Q$. Axiom C7, a (classical) contrapositive of the classical axiom, is significantly stronger, since the condition $P \notin l$ is an affirmative condition.

Heyting and van Dalen have used an apparently weaker version of Axiom C7; it is Heyting's Axiom VI [H28], and van Dalen's Lemma 3(f), obtained using his axiom Ax5 [vDal96]. This weaker version states: "If l and m are distinct lines, P is a point such that $P \neq l \cdot m$, and $P \in l$, then $P \notin m$." However, it is easily shown that the two versions are equivalent.

Axiom C4 states that at least three distinct points lie on any given line; this is the usual classical axiom. Then, for the study of projectivities, Axiom E is added, increasing the required number of points to six. Recently, a constructive proof in [M18], of an essential result concerning harmonic conjugates, required at least eight points on a line; thus we have:

Problem. Determine the minimum number of points on a line that are required for the various constructive proofs concerning the projective plane $\mathbb{P}$. Examine the propositions involved for the exceptional small finite planes.

The axioms and definitions of constructive projective geometry can be given a variety of different arrangements. For example, in [vDal96] the outside relation $P \notin l$ is taken as a primitive notion, and the condition of Axiom C6 above becomes the definition of the incidence relation $P \in l$. See also [H28, Pam05, Pam11, vPla95].

The axiom system could be extended; thus we have:

Problem. Extend the constructive axiom group C to projective space, and derive constructive versions of the main classical theorems.

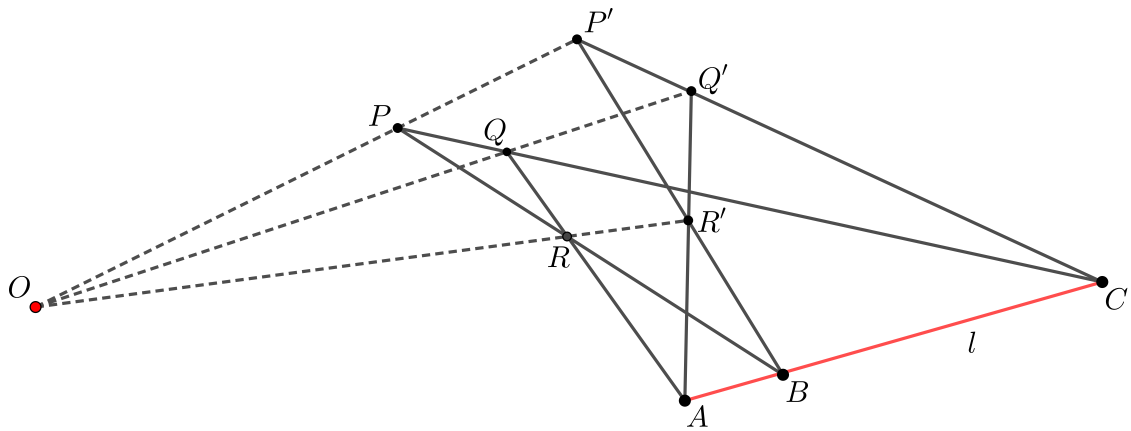
2.2 Desargues's Theorem

Desargues's Theorem is assumed as an axiom; the converse is then proved as a consequence.

Two triangles are *distinct* if corresponding vertices are distinct and corresponding sides are distinct; it is then easily shown that the three lines joining corresponding vertices are distinct, and the three points of intersection of corresponding sides are distinct. Distinct triangles are said to be *perspective from the center O* if the lines joining corresponding vertices are concurrent at the point O, and O lies outside each of the six sides. Distinct triangles are said to be *perspective from the axis l* if the points of intersection of corresponding sides are collinear on the line l, and l avoids each of the six vertices.

Axiom D. Desargues's Theorem. *If two triangles are perspective from a center, then they are perspective from an axis.*

The proof of the converse is included below, as an example of constructive methods in geometry.



Theorem. Converse of Desargues's Theorem. *If two triangles are perspective from an axis, then they are perspective from a center.* [M16, Thm. 3.2]

Proof. We are given distinct triangles PQR and $P'Q'R'$, with collinear points of intersection of corresponding sides, $A = QR \cdot Q'R'$, $B = PR \cdot P'R'$, $C = PQ \cdot P'Q'$, and with all six vertices lying outside the axis $l = AB$. Set $O = PP' \cdot QQ'$.

The points A, Q, Q' are distinct, and the points B, P, P' are distinct. Since $Q \neq A = QR \cdot Q'R'$, it follows from Axiom C7 that $Q \notin Q'R' = AQ'$; thus the points A, Q, Q' are noncollinear, and similarly for B, P, P'. Since $P \notin AB$, we have $AB \neq BP$. Since $A \neq B = AB \cdot BP$, it follows that $A \notin BP$, so $AQ \neq BP$. By symmetry, $AQ' \neq BP'$. This shows that the triangles AQQ', BPP' are distinct.

The lines AB, PQ, $P'Q'$, joining corresponding vertices of the triangles AQQ', BPP', are concurrent at C. Since $Q \notin AB$, we have $AB \neq AQ$. From $C \neq A = AB \cdot AQ$, it follows that $C \notin AQ$; by symmetry, $C \notin AQ'$. Since $Q' \neq C = PQ \cdot P'Q'$, it follows that $Q' \notin PQ$; thus $QQ' \neq PQ$, i.e., $CQ \neq QQ'$. From $C \neq Q = CQ \cdot QQ'$, we have $C \notin QQ'$. Thus C lies outside each side of triangle AQQ', and similarly for triangle BPP'. Thus the triangles AQQ', BPP' are perspective from the center C.

It follows from Axiom D that the triangles AQQ', BPP' are perspective from the axis $(AQ \cdot BP)(AQ' \cdot BP') = RR'$, the axis avoids all six vertices, and $O \in RR'$. Thus the lines PP', QQ', RR', joining corresponding vertices of the given triangles, are concurrent at O. Since $Q \notin RR'$, we have $Q \neq O$. From $O \neq Q = QQ' \cdot PQ$, it follows that $O \notin PQ$. By symmetry, O lies outside each side of the given triangles.

Hence the triangles PQR and $P'Q'R'$ are perspective from the center O. $\square$

2.3 Duality

Given any statement, the *dual statement* is obtained by interchanging the words "point" and "line". For example:

Dual of Axiom C5. *For any points P and Q on the projective plane $\mathbb{P}$, if there exists a line l such that $P \in l$, and $Q \notin l$, then $P \neq Q$.*

Dual of Axiom C7. *Let A and B be distinct points on the projective plane $\mathbb{P}$. If l is a line such that $l \neq AB$, then either $A \notin l$ or $B \notin l$.*

Clearly, Axiom C6 is self-dual. *Duality* in a given system is the principle that the dual of any true statement is also true. Duality of the construction of the plane $\mathbb{P}$, and of the axiom system, is verified as follows:

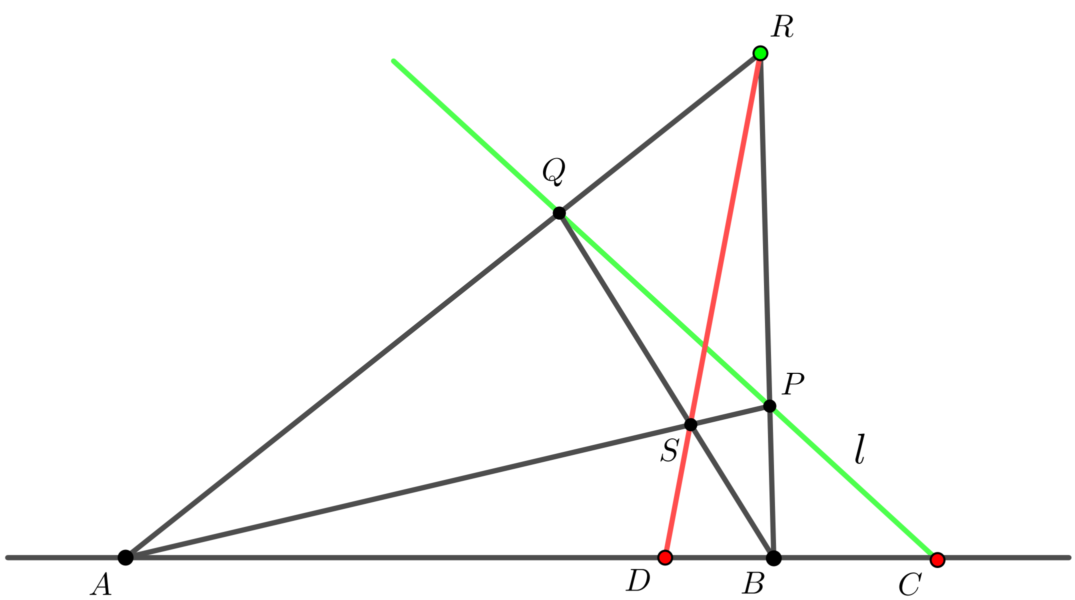
Theorem. *The definition of the projective plane $\mathbb{P}$ is self-dual. The dual of each axiom in axiom group C is valid on $\mathbb{P}$.* [M16, Thm. 2.10]

The dual of the definition of the outside relation $P \notin l$ is also verified:

Theorem. *Let P be any point on the projective plane $\mathbb{P}$, and l any line. Then $P \notin l$ if and only if $l \neq m$ for any line m that passes through P.* [M16, Thm. 2.11]

3 Harmonic conjugates

In the construction of the projective plane $\mathbb{P}$, harmonic conjugates have an essential role, with applications to projectivities, involutions, and polarity. In the drawing below, the quadrangle $PQRS$, which is often used classically, appears to determine the harmonic conjugate D of the point C, with respect to the base points A and B. However, this is only valid when C is distinct from each base point; thus we must use a definition that applies to every point on the base line AB.



Definition. Let A and B be distinct points on the projective plane $\mathbb{P}$. For any point C on the line AB, select a line l through C, distinct from AB, and select a point R lying outside each of the lines AB and l. Set $P = BR \cdot l$, $Q = AR \cdot l$, and $S = AP \cdot BQ$. The point $D = AB \cdot RS$ is called the *harmonic conjugate of C with respect to the points A, B*; we write $D = h(A, B; C)$. [M16, Defn. 4.1]

Since the construction of a harmonic conjugate requires the selection of auxiliary elements, it must be verified that the result is independent of the choice of these auxiliary elements. The proof given in [M16] for the invariance theorem is incorrect; apart from the error, the proof there is excessively complicated, and objectionable on several counts. A correct proof appears in a later paper.

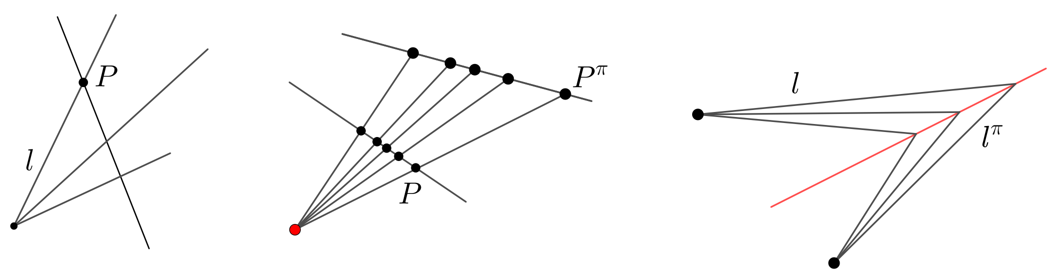
Theorem. Invariance Theorem. *Let C be any point on the line AB, and let auxiliary element selections (l, R) and (l', R') be used to construct harmonic conjugates D and D' of the point C. Then $D = D'$; the harmonic conjugate construction is independent of the choice of auxiliary elements.* [M18, Thm. 3.2]

In the special case of a point distinct from both base points, constructive harmonic conjugates can be related to the traditional quadrangle configuration, due to Philippe de La Hire (1640 – 1718):

Corollary. *Let A, B, C, D be collinear points, with $A \neq B$, and C distinct from each of the points A and B. Then $D = h(A, B; C)$ if and only if there exists a quadrangle with vertices outside the line AB, of which two opposite sides intersect at A, two other opposite sides intersect at B, while the remaining two sides meet the base line AB at C and D.* [M18, Cor. 3.3]

4 Projectivities

The elementary mappings of a projective plane are *sections,* bijections relating a pencil of lines with a range of points. Certain combinations of sections result in *projections,* mapping a range of points onto another range, projecting from a center, or mapping a pencil of lines onto another pencil, projecting from an axis. These sections and projections are the *perspectivities* of the plane.



The product (composition) of two perspectivities need not be a perspectivity. For the projective plane $\mathbb{P}$, a finite product of perspectivities is called a *projectivity;* this is the definition used by Jean-Victor Poncelet (1788 – 1867) [Pon22]. Subsequently, Karl Georg Christian von Staudt (1798 – 1867) [vSta47] defined a *projectivity* as a mapping of a range or a pencil that preserves harmonic conjugates. Classically, the two notions of perspectivity are equivalent. Constructively, we have:

Theorem. *A projectivity of the projective plane $\mathbb{P}$ preserves harmonic conjugates. Thus every Poncelet projectivity is a von Staudt projectivity.* [M16, Thm. 5.3]

However, the constructive content of the converse is not known; thus we have:

Problem. On the projective plane $\mathbb{P}$, show that every von Staudt projectivity is a Poncelet projectivity, or construct a counterexample.

It is necessary to establish the existence of projectivities:

Theorem. *Given any three distinct points P, Q, R in a range $\overline{l}$, and any three distinct points P', Q', R' in a range $\overline{m}$, there exists a projectivity $\pi : \overline{l} \to \overline{m}$ such that the points P,Q,R map into the points P', Q', R', in the order given.* [M16, Thm. 5.6]

Classically, the projectivity produced by this theorem is the product of at most three perspectivities. However, the constructive proof in [M16] requires six perspectivities; thus we have:

Problem. Determine the minimum number of perspectivities required for the above theorem.

A projectivity π of order 2 (π^2 is the identity ι) is called an *involution*; this term was first used by Girard Desargues (1591-1661). In [Des64], Desargues introduced seventy new geometric terms; they were considered highly unusual, and met with sharp criticism and ridicule by his contemporaries. Of these seventy terms, *involution* is the only one to have survived. One example of an involution is the harmonic conjugate relation:

Theorem. *Let A and B be distinct points in a range $\overline{l}$, and let υ be the mapping of harmonic conjugacy with respect to the base points A, B; i.e., set $X^\upsilon = h(A, B; X)$, for all points X in the range $\overline{l}$. Then υ is an involution.* [M16, Thm. 7.2]

5 Fundamental Theorem

The fundamental theorem of projective geometry [vSta47] is required for many results, including Pascal's Theorem. Classically, the fundamental theorem is derived from axioms of order and continuity. For the projective plane $\mathbb{P}$, since no axioms of order and continuity have been adopted, the crucial component of the fundamental theorem must be derived directly from an axiom:

Axiom T. *If a projectivity π of a range or pencil onto itself has three distinct fixed elements, then it is the identity ι.*

Classically, Axiom T is often given the following equivalent form: *Let π be a projectivity from a range onto itself, with $\pi \neq \iota$, and distinct fixed points M and N. If Q is a point of the range distinct from both M and N, then $Q^\pi \neq Q$.* Constructively, this appears to be a stronger statement, since the implication "$\neg(Q^\pi = Q)$ implies $Q^\pi \neq Q$" is constructively invalid; thus we have:

Problem. Give a proof of the apparently-stronger, alternative statement for Axiom T, or show that it is constructively stronger.

To prove that the alternative statement is constructively stronger would require a Brouwerian counterexample. To determine the specific nonconstructivities in a classical theory, and thereby to indicate feasible directions for constructive work, Brouwerian counterexamples are used, in conjunction with nonconstructive omniscience principles. A *Brouwerian counterexample* is a proof that a given statement implies an omniscience principle. In turn, an *omniscience principle* would imply solutions or significant information for a large number of well-known unsolved problems. This method was introduced by L. E. J. Brouwer [Brou08] to demonstrate that use of the law of excluded middle inhibits mathematics from attaining its full significance. A statement is considered *constructively invalid* if it implies an omniscience principle. The omniscience principles can be expressed in terms of real numbers; the following are most often utilized:

Limited principle of omniscience (LPO). *For any real number α, either $\alpha = 0$ or $\alpha \neq 0$.*

Weak limited principle of omniscience (WLPO). *For any real number α, either $\alpha = 0$ or $\neg(\alpha = 0)$.*

Lesser limited principle of omniscience (LLPO). *For any real number α, either $\alpha \leq 0$ or $\alpha \geq 0$.*

Markov's principle. *For any real number α, if $\neg(\alpha = 0)$, then $\alpha \neq 0$.*

For work according to Bishop-type strict constructivism, as followed here, these principles, consequences of the law of excluded middle, are used only to demonstrate the nonconstructive nature of certain classical statements, and are not accepted for developing a constructive theory. Markov's Principle, however, is used for work in recursive function theory.

Theorem. Fundamental Theorem. *Given any three distinct points P,Q,R in a range $\overline{l}$, and any three distinct points P', Q', R' in a range $\overline{m}$, there exists a unique projectivity $\pi : \overline{l} \to \overline{m}$ such that the points P,Q,R map into the points P', Q', R', in the order given.* [M16, Thm. 6.1]

Proof. The existence of the required projectivity is provided by the second theorem in Section 4 above. Uniqueness, however, requires Axiom T. □

Classically, the fundamental theorem is derived from axioms of order and continuity; thus we have:

Problem. Introduce constructive axioms of order and continuity for the projective plane $\mathbb{P}$; derive Axiom T and the fundamental theorem.

It follows from the fundamental theorem that any projectivity between distinct ranges, or between distinct pencils, that has a fixed element is a perspectivity [M16, Cor. 6.2]. A projectivity π such that $x^\pi \neq x$, for all elements x, is called *nonperspective.*

The concept of projectivity is extended to the entire plane. A *collineation* of the projective plane $\mathbb{P}$ is a bijection of the family $\mathscr{P}$ of points, onto itself, that preserves collinearity and noncollinearity. A collineation σ induces an analogous bijection σ' of the family $\mathscr{L}$ of lines. A collineation is *projective* if it induces a projectivity on each range and each pencil of the plane.

The following theorem is a constructivization of one of the main results in the classical theory.

Theorem. *A projective collineation with four distinct fixed points, each three of which are noncollinear, is the identity.* [M16, Prop. 6.7]

Proof. Let the collineation σ have the fixed points P, Q, R, S as specified; thus the three distinct lines PQ, PR, PS are fixed. The mapping σ' induces a projectivity on the pencil P^*; by the fundamental theorem, this projectivity is the identity. Thus every line through P is fixed under σ'; similarly, the same is true for the other three points.

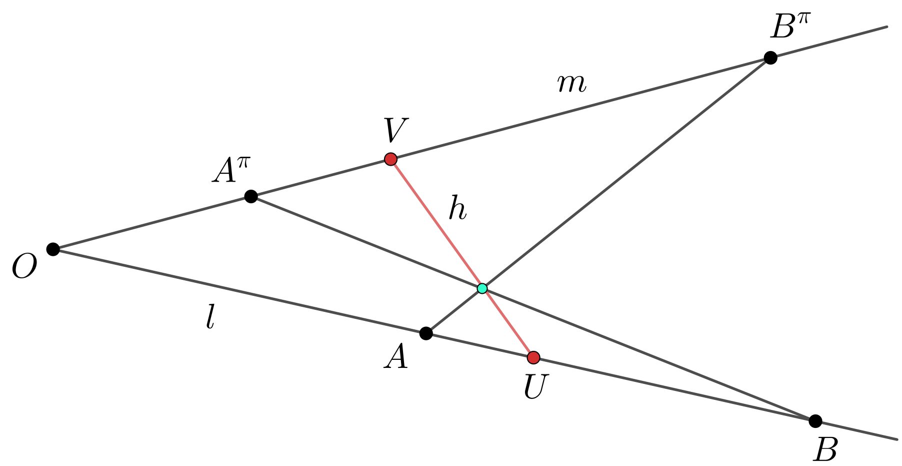
Now let X be any point on the plane. By three successive applications of cotransitivity for points, we may assume that X is distinct from each of the points P, Q, R. Since $PQ \neq PR$, using cotransitivity for lines we may assume that $XP \neq PQ$. Since $Q \neq P = XP \cdot PQ$, it follows from Axiom C7 that $Q \notin XP$, and thus $XP \neq XQ$. Since $X = XP \cdot XQ$, and the lines XP and XQ are fixed under σ', it follows that $\sigma X = X$. $\square$

Problem. The above theorem ensures the uniqueness of a collineation that maps four distinct points, each three of which are noncollinear, into four distinct specified points, each three of which are also noncollinear. Establish the existence of such a collineation for the projective plane $\mathbb{P}$.

The classical theory of the axis of homology has also been constructivized.

Definition. Let $\pi : \bar{l} \to \overline{m}$ be a nonperspective projectivity between distinct ranges on the projective plane $\mathbb{P}$. Set $O = l \cdot m$, $V = O^\pi$, and $U = O^{\pi^{-1}}$; then the line $h = UV$ is called the *axis of homology for* π. [M16, Defn. 6.4]

The following theorem is the main result concerning the axis of homology; the proof requires the fundamental theorem.



Theorem. *Let $\pi : \bar{l} \to \overline{m}$ be a nonperspective projectivity between distinct ranges on the projective plane $\mathbb{P}$. If A and B are distinct points on l, each distinct from the common point O, then the point $AB^\pi \cdot BA^\pi$ lies on the axis of homology h.* [M16, Thm. 6.5]

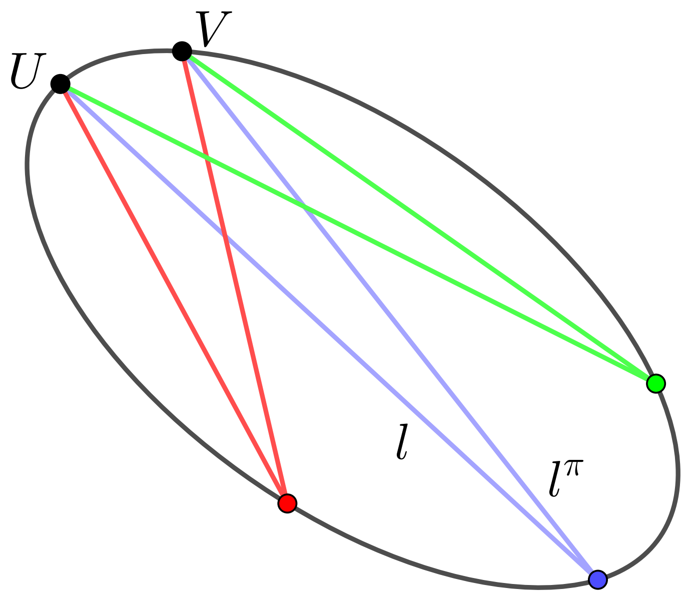
6 Conics

The conic sections have a long history; they were discovered by Menaechmus (ca. 340 BC) and studied by the Greek geometers to the time of Pappus of Alexandria (ca. 320 AD). The motivation for Menaechmus's discovery was a geometrical problem, put forth by the oracle on the island of Delos, the solution of which would have provided a remedy for the Athenian plague of 430 BC. Unfortunately, Menaechmus's solution was too late; see [Cox55, p. 79] for details.

In the 17th century, an intense new interest in the conics arose in connection with projective geometry. On a projective plane there is no distinction between the hyperbola, parabola, and ellipse; these arise only in the affine plane after a line at infinity is removed. Which of the three forms results depends on whether that line meets the conic at two, one, or no points.

6.1 Construction of a conic

Conics on the projective plane $\mathbb{P}$ are defined by means of projectivities, using the method of Jakob Steiner (1796 – 1863) [Ste32]. Alternatively, in classical works conics are often defined by means of polarities, using the method of von Staudt; see the problem stated at the end of Section 7 below.



Definition. (Steiner) Let $\pi : U^* \to V^*$ be a nonperspective projectivity between distinct pencils of lines on the projective plane $\mathbb{P}$. The *conic* $\kappa = \kappa(\pi; U, V)$ *defined by* π is the locus of points $\{l \cdot l^\pi : l \in U^*\}$. For any point X on $\mathbb{P}$, we will say that X *lies outside* κ, written $X \notin \kappa$, if $X \neq Y$ for all points Y on κ. [M16, Defn. 8.1]

Problem. This definition, with the assumption that the given projectivity is nonperspective, produces what is usually called a *non-singular* conic. Singular conics await constructive investigation.

6.2 Properties

The next theorem establishes an essential property of a conic, an analogue of the tightness property for inequalities; it can be viewed as an extension of Axiom C6: "If $\neg(P \notin l)$, then $P \in l$."

Theorem. *Let $\kappa = \kappa(\pi; U, V)$ be a conic on the projective plane $\mathbb{P}$. For any point X on $\mathbb{P}$, if $\neg(X \notin \kappa)$, then $X \in \kappa$.* [M16, Prop. 8.2(d)]

Proof. Let X be a point on the plane such that $\neg(X \notin \kappa)$. By cotransitivity and symmetry, we may assume that $X \neq U$. Set $z = UX$; then $Z = z \cdot z^\pi$ is a point of κ. Suppose that $X \neq Z$.

We now show that $X \neq Y$ for any point Y of κ. Either $Y \neq X$ or $Y \neq U$. We need to consider only the second case; set $y = UY$, it follows that $Y = y \cdot y^\pi$. Either $Y \neq X$ or $Y \neq Z$; again, it suffices to consider the second case. Since $Y \neq Z = z \cdot z^\pi$, it follows from Axiom C7 that either $Y \notin z$ or $Y \notin z^\pi$. In the first subcase, $y \neq z$. In the second subcase, $y^\pi \neq z^\pi$, and since π is a bijection we again have $y \neq z$. Since $X \neq U = y \cdot z$, it follows that $X \notin y$, and thus $X \neq Y$.

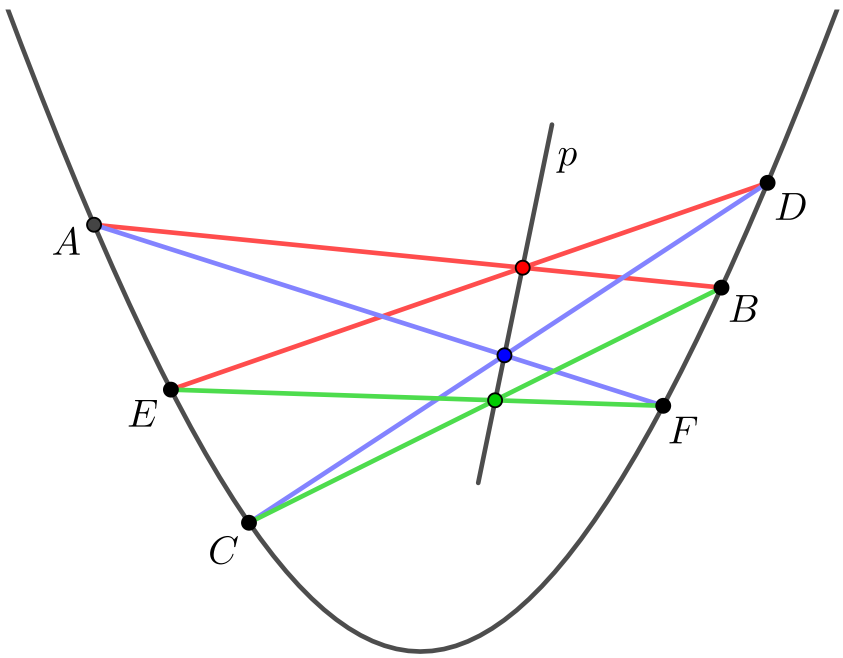
The above shows that $X \notin \kappa$, contradicting the hypothesis. It follows that $X = Z$, and hence $X \in \kappa$. $\square$

Using this theorem and other preliminary results, many well-known classical results are obtained constructively; for example, the following basic result:

Theorem. *There exists a unique conic containing any given five distinct points, each three of which are noncollinear.* [M16, Prop. 8.3]

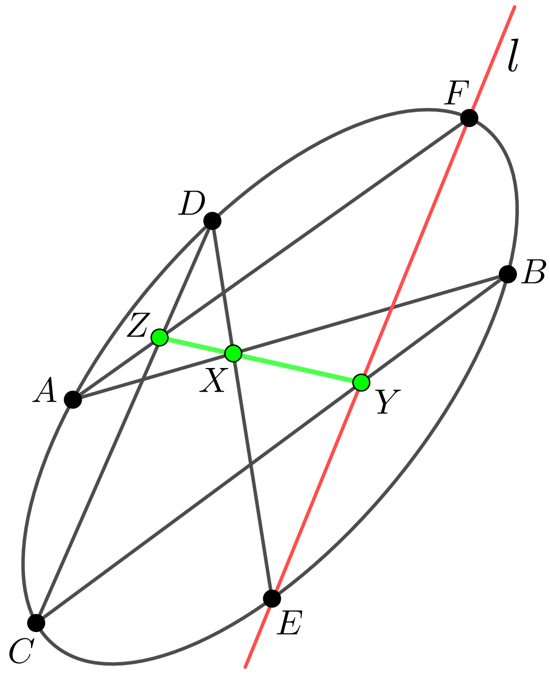
6.3 Pascal's Theorem

Perhaps the most widely-known classical result concerning conics is the following, due to Blaise Pascal (1623-1662) [Pas39]; it also has a constructive proof.



Theorem. PASCAL'S THEOREM. *Let a simple hexagon $ABCDEF$ be inscribed in a conic κ. Then the three points of intersection of the pairs of opposite sides are distinct and collinear.* [M16, Thm. 9.2]

According to legend, Pascal gave in addition some four hundred corollaries. Only one has been constructivized, it recalls a traditional construction method for drawing a conic "point by point" on paper; for example, as in [Y30, p. 68].



Corollary. *Let A, B, C, D, E be five distinct points of a conic κ. If l is a line through E that avoids each of the other four points, and l passes through a distinct sixth point F of κ, then*

$$F = l \cdot A(CD \cdot (AB \cdot DE)(BC \cdot l)).$$

[M16, Cor. 9.3]

Proof. The *Pascal line* p of the hexagon $ABCDEF$ passes through the three distinct points $X = AB \cdot DE$, $Y = BC \cdot EF$, and $Z = CD \cdot AF$. Since $A \notin CD$, we have $A \neq Z$, and it follows that $AF = AZ$. Since $B \notin CD$, we have $BC \neq CD$, so by cotransitivity for lines either $p \neq BC$ or $p \neq CD$. In the first case, since $C \notin EF$, we have $C \neq Y = BC \cdot p$, and it follows from Axiom C7 that $C \notin p$. Thus in both cases we have $CD \neq p$, and $Z = CD \cdot p$. Finally,

$$\begin{aligned} F &= EF \cdot AF = l \cdot AZ = l \cdot A(CD \cdot p) \\ &= l \cdot A(CD \cdot XY) = l \cdot A(CD \cdot (AB \cdot DE)(BC \cdot l)) \end{aligned}$$

□

7 Polarity

The role of symmetry in projective geometry reaches a peak of elegance in the theory of polarity, introduced by von Staudt in 1847.

A *correlation* is a mapping of the points of the projective plane to the lines, together with a mapping of the lines to the points, that preserves collinearity and concurrence. The correlation is *involutory* if it is of order 2, and is then called a *polarity.* A conic determines a polarity; each point of the plane has a corresponding polar, and each line has a corresponding pole.

7.1 Tangents and secants

The construction of poles and polars determined by a conic is dependent upon the existence of tangents and secants. A line t that passes through a point P on a conic κ is said to be *tangent to κ at P* if P is the unique point of κ that lies on t. A line that passes through two distinct points of a conic κ is a *secant of κ*. For the construction of poles and polars, it has been necessary to adopt an additional axiom:

Axiom P. The tangents at any three distinct points of a conic are nonconcurrent.

Problem. Determine whether this axiom can be derived from the others.

The tangents and secants to a conic are related by means of projectivities; the tangent at a point on a conic is the projective image of any secant through the point:

Theorem. *Let κ be a conic on the projective plane $\mathbb{P}$, P a point on κ, and t a line passing through P. The line t is tangent to κ at P if and only if for any point Q of κ with $Q \neq P$, if s is the secant QP, and π is the nonperspective projectivity such that $\kappa = \kappa(\pi; Q, P)$, then $t = s^\pi$.* [M16, Prop. 10.2(b)]

This theorem ensures the existence of tangents. To establish the existence of secants, it is first shown that a line through a point on a conic, if not the tangent, is a secant:

Lemma. *Let κ be a conic on the projective plane $\mathbb{P}$, P a point on κ, and t the tangent to κ at P. If l is a line passing through P, and $l \neq t$, then l passes through a second point R of κ, distinct from P; thus l is a secant of κ.* [M16, Lm. 10.9]

Using this lemma, the next theorem will provide the secants needed for the following results. The need for this theorem contrasts with complex geometry, where every line meets every conic.

Theorem. *Let κ be a conic on the projective plane $\mathbb{P}$. Through any given point P of the plane, at least two distinct secants of κ can be constructed.* [M16, Thm. 10.10(a)]

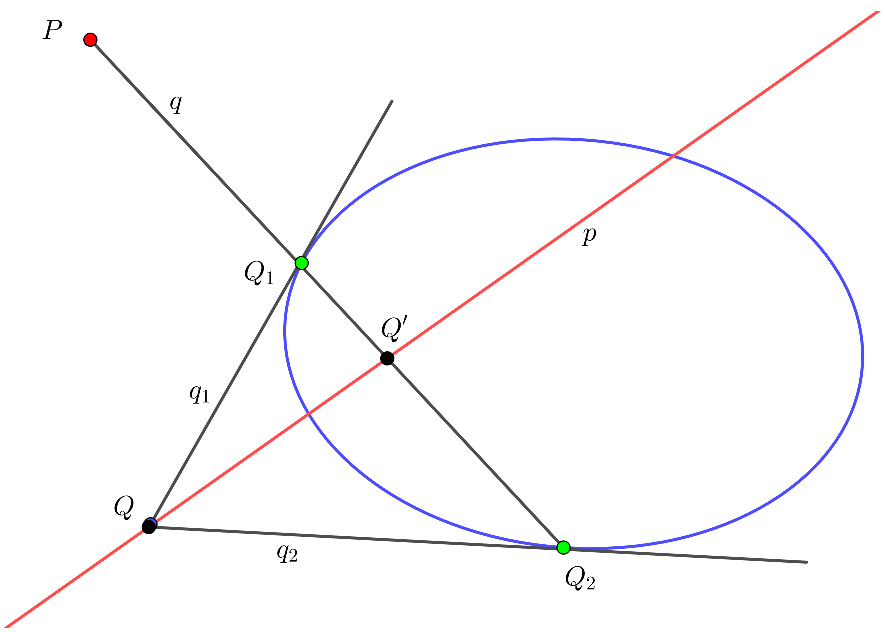
Proof. Select distinct points A, B, C on κ, with tangents a, b, c. By Axiom P, these tangents are nonconcurrent; thus the points $E = a \cdot b$ and $F = b \cdot c$ are distinct. Either $P \neq E$ or $P \neq F$; it suffices to consider the first case. By Axiom C7, either $P \notin a$ or $P \notin b$. It suffices to consider the first subcase; thus $P \neq A$ and $PA \neq a$. It follows from the lemma that PA is a secant.

Denote the second point of PA that lies on κ by R, and choose distinct points A', B', C' on κ, each distinct from both A and R. With these three points, construct a secant through P using the above method; we may assume that it is PA'. Since $A' \notin AR = PA$, it follows that $PA' \neq PA$. $\square$

7.2 Construction of polars and poles

The traditional method for defining a polar uses an inscribed quadrangle, and must consider separately points on or outside a conic. Constructively, this method is not available; thus polars are constructed by means of harmonic conjugates.

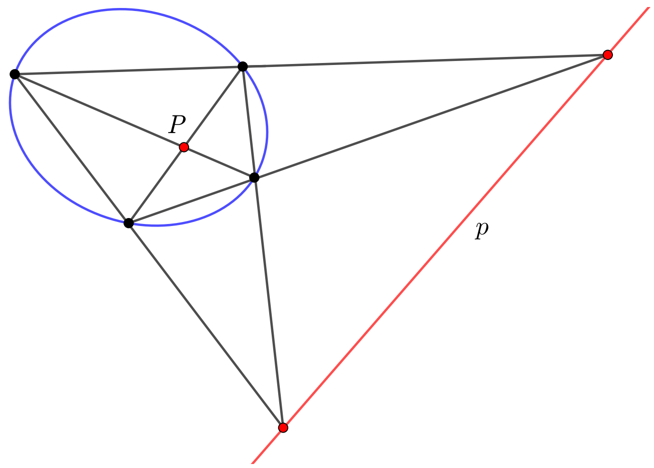
The discussion of harmonic conjugates in Section 3 included an invariance theorem to show that the result of the construction is independent of the selection of auxiliary elements. Now, the definition of the polar of a point must be shown to be independent of the choice of an auxiliary secant; the proof requires the invariance theorem for harmonic conjugates.



Theorem. Construction of a polar. *Let κ be a conic on the projective plane $\mathbb{P}$, and let P be any point on the plane. Through the point P, construct a secant q of κ. Denote the intersections of q with κ by Q_1 and Q_2, and let the tangents at these points be denoted q_1 and q_2. Set $Q = q_1 \cdot q_2$. Set $Q' = h(Q_1, Q_2; P)$, the harmonic conjugate of P with respect to the base points Q_1, Q_2. Then the line $p = QQ'$ is independent of the choice of the secant q.* [M16, Thm. 11.1]

Definition. Let κ be a conic on the projective plane $\mathbb{P}$, and let P be any point on the plane. The line $p = QQ'$ in the above theorem is called the *polar of P with respect to κ.* [M16, Defn. 11.2]

Note that if P lies on κ, then the polar of P is the tangent to κ at P. The corollary below will relate this constructive theory of polars to a classical construction that uses quadrangles. The three *diagonal points* of a quadrangle are the intersection points of the three pairs of opposite sides. We adopt Fano's Axiom: *The diagonal points of any quadrangle are noncollinear.* Gino Fano (1871-1952) studied finite projective planes, some of which do not satisfy Fano's Axiom.



Corollary. *Let κ be a conic on the projective plane $\mathbb{P}$, and let P be any point outside κ. Inscribe a quadrangle in κ with P as one diagonal point. Then the polar of P is the line p joining the other two diagonal points.* [M16, Cor. 11.4]

Any three distinct points on a conic are noncollinear [M16, Prop. 8.2(b)]. Thus, if P is any diagonal point of a quadrangle inscribed in a conic κ, it follows that $\neg(P \in \kappa)$. However, it does not immediately follow that P lies outside κ; thus we have:

Problem. If κ is a conic on the projective plane $\mathbb{P}$, and P is a diagonal point of a quadrangle inscribed in κ, show that $P \notin \kappa$.

Definition. Let κ be a conic on the projective plane $\mathbb{P}$, and l any line on $\mathbb{P}$. A construction analogous to that of the above theorem results in a point L, called the *pole of l with respect to κ*. [M16, Defn. 11.5]

The following theorem shows that any conic on the plane $\mathbb{P}$ determines a polarity:

Theorem. *Let κ be a conic on the projective plane $\mathbb{P}$. If the line p is the polar of the point P, then P is the pole of p, and conversely.* [M16, Thm. 11.6(a)]

The definition of conic in Section 6 used the Steiner method [Ste32], with projectivities. Later, von Staudt [vSta47] defined a conic by means of a polarity: a point lies on the conic if its polar passes through the point. Classically, the two definitions produce the same conics; thus we have:

Problem. Construct correlations and polarities based on the axioms for the projective plane $\mathbb{P}$, develop the theory of conics constructively using the von Staudt definition, and prove that von Staudt conics are equivalent to the Steiner conics constructed in Section 6.1 above.

8 Consistency of the axiom system

The consistency of the axiom system for the synthetic projective plane $\mathbb{P}$ is established by an analytic model. A projective plane $\mathbb{P}^2(\mathbb{R})$ is built from subspaces of the linear space $\mathbb{R}^3$, using only constructive properties of the real numbers. The axioms adopted for the synthetic plane $\mathbb{P}$ have been chosen to reflect the properties of the analytic plane $\mathbb{P}^2(\mathbb{R})$, taking note of Bishop's thesis, "All mathematics should have numerical meaning" [B67, p. ix; BB85, p. 3].

The model is built following well-known classical methods, adding constructive refinements to the definitions and proofs. The analytic plane $\mathbb{P}^2(\mathbb{R})$ consists of a family $\mathscr{P}_2$ of points, and a family $\mathscr{L}_2$ of lines; a *point* P in $\mathscr{P}_2$ is a subspace of dimension 1 of the linear space $\mathbb{R}^3$, a *line* λ in $\mathscr{L}_2$ is a subspace of dimension 2. The inequality relations, the incidence relation, and the outside relation are defined by means of vector operations. All the essential axioms adopted for the synthetic plane $\mathbb{P}$, and all the required properties, such as cotransitivity, tightness, and duality, are verified.

Theorem. *Axiom group C, and Axioms F, D, E, T, are valid on the analytic projective plane* $\mathbb{P}^2(\mathbb{R})$. [M16, Thm. 14.2]

The Brouwerian counterexample below shows that on the plane $\mathbb{P}^2(\mathbb{R})$ the validity of Axiom C3, which ensures the existence of a common point for any two distinct lines, is dependent on the restriction to distinct lines. By duality, the two statements of the example are equivalent. The proof of the first statement is easier to visualize, and can be described informally as follows: On $\mathbb{R}^2$, thought of as a portion of $\mathbb{P}^2(\mathbb{R})$, consider two points which are extremely near or at the origin, with P on the x-axis, and Q on the y-axis. If P is very slightly off the origin, and Q is at the origin, then the x-axis is the required line. In the opposite situation, the y-axis would be required. In any conceivable constructive routine, such a large change in the output, resulting from a minuscule variation of the input, would reveal a severe discontinuity, and a strong indication that the statement in question is constructively invalid.

Example. On the analytic projective plane $\mathbb{P}^2(\mathbb{R})$, the following statements are constructively invalid:

(i) *Given any points P and Q, there exists a line that passes through both points.*

(ii) *Given any lines λ and μ, there exists a point that lies on both lines.*

[M16, Example 14.1]

Proof. By duality, it will suffice to consider the second statement. When the non-zero vector $t = (t_1, t_2, t_3)$ spans the point T in $\mathbb{P}^2(\mathbb{R})$, we write $T = \langle t \rangle = \langle t_1, t_2, t_3 \rangle$. When the vectors u, v span the line λ, and $w = u \times v$, we write $\lambda = [w] = [w_1, w_2, w_3]$. The relation $T \in \lambda$ is defined by the inner product, $t \cdot w = 0$.

Let α be any real number, and set $\alpha^+ = \max\{\alpha, 0\}$, and $\alpha^- = \max\{-\alpha, 0\}$. Define lines $\lambda = [\alpha^+, 0, 1]$ and $\mu = [0, \alpha^-, 1]$. By hypothesis, we have a point $T = \langle t \rangle = \langle t_1, t_2, t_3 \rangle$ that lies on both lines. Thus $\alpha^+ t_1 + t_3 = 0$, and $\alpha^- t_2 + t_3 = 0$. If $t_3 \neq 0$, then we have both $\alpha^+ \neq 0$ and $\alpha^- \neq 0$, an absurdity; thus $t_3 = 0$. This leaves two cases. If $t_1 \neq 0$, then $\alpha^+ = 0$, so $\alpha \leq 0$, while if $t_2 \neq 0$, then $\alpha^- = 0$, so $\alpha \geq 0$. Hence LLPO results. $\square$

Problem. Develop the theory of conics for the analytic plane $\mathbb{P}^2(\mathbb{R})$; compare the results with those for the synthetic plane $\mathbb{P}$. On the plane $\mathbb{P}^2(\mathbb{R})$, determine the constructive validity of Axiom P of Section 7 above.

Problem. For the analytic projective plane $\mathbb{P}^2(\mathbb{R})$, apply constructive methods to the study of harmonic conjugates, cross ratios, and other topics of classical projective geometry.

Problem. Although the model $\mathbb{P}^2(\mathbb{R})$ establishes the consistency of the axiom system used for the projective plane $\mathbb{P}$, it remains to prove the independence of the axiom system, or to reduce it to an independent system.

Part II

Projective extensions

The notion of infinity has mystified finite humans for millennia. On the analytic projective plane $\mathbb{P}^2(\mathbb{R})$, where points and lines are merely lines and planes through the origin in $\mathbb{R}^3$, it is no surprise to notice that any two distinct lines meet at a unique point. However, to envision two parallel lines on $\mathbb{R}^2$ meeting at infinity requires some imagination. Johannes Kepler (1571 – 1630) invented the term "focus" in regard to ellipses, and stated that a parabola also has two foci, with one at infinity. This idea was extended by Poncelet, leading to the concepts of a line at infinity, and a projective plane.

In the classical theory, a projective extension of an affine plane is a fairly simple matter: each pencil of parallel lines determines a point at infinity, at which the lines meet, and these points form the line at infinity. A projective plane results, and the required projective axioms are satisfied. The extension of the metric plane $\mathbb{R}^2$ to a projective plane is often described heuristically, with lamps and shadows; see, for example, [Cox55, Section 1.3].

There have been at least three constructive attempts to extend an affine plane to a projective plane. An extension by A. Heyting [H59] uses elements called "projective points" and "projective lines". The extension constructed in [M14] uses elements called "prime pencils" and "virtual lines", resulting in a projective plane with different properties. The analytic projective plane $\mathbb{P}^2(\mathbb{R})$ discussed in Section 8 above, constructed using subspaces of $\mathbb{R}^3$, can be viewed as an extension of the metric plane $\mathbb{R}^2$; it also has distinctive properties.

The differences between these several extensions involve the crucial axiom concerning the existence of a point common to two lines, and the cotransitivity property. The statement that any two distinct lines have a common point is called the *Common Point Property (CPP),* while the *Strong Common Point Property (SCPP)* is the same statement without the restriction to distinct lines. The analytic extension $\mathbb{P}^2(\mathbb{R})$ of $\mathbb{R}^2$ satisfies both CPP and cotransitivity, but *not* SCPP. Neither synthetic extension satisfies both cotransitivity and CPP. The Heyting extension satisfies cotransitivity, but the essential axiom CPP has not been verified. On the virtual line extension, CPP is satisfied, and even SCPP; however, cotransitivity is constructively invalid, and this is now seen as a serious limitation. The analytic extension $\mathbb{P}^2(\mathbb{R})$ could be taken as a standard; one might demand that the basic properties of $\mathbb{P}^2(\mathbb{R})$ hold in any acceptable synthetic extension, and then neither of the two synthetic extensions would suffice.

Problem. Construct a synthetic projective extension of an affine plane which has the usual properties of a projective plane, including both the common point property and cotransitivity.

9 Heyting extension

In [H59], A. Heyting adopts axioms for both affine and projective geometry. Then, from a plane affine geometry $(\mathscr{P}, \mathscr{L})$, Heyting constructs an extension (Π, Λ), consisting of *projective points* of the form

$$\mathfrak{P}(l, m) = \{n \in \mathscr{L} \,:\, n \cap l = l \cap m \text{ or } n \cap m = l \cap m\}$$

where $l, m \in \mathscr{L}$ with $l \neq m$, and *projective lines* of the form

$$\lambda(\mathfrak{A}, \mathfrak{B}) = \{\mathfrak{Q} \in \Pi \,:\, \mathfrak{Q} \cap \mathfrak{A} = \mathfrak{A} \cap \mathfrak{B} \text{ or } \mathfrak{Q} \cap \mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}\}$$

where $\mathfrak{A}, \mathfrak{B} \in \Pi$ with $\mathfrak{A} \neq \mathfrak{B}$.

With the Heyting definition of projective point, if the original two lines l and m intersect, then $\mathfrak{P}(l, m)$ is the pencil of all lines passing through the point of intersection, while if the lines are parallel, then $\mathfrak{P}(l, m)$ is the pencil of all lines parallel to the original two. In these cases, the definition determines either a finite point of the extension, or a point on the line at infinity. More significant is the fact that even when the status of the two original lines is not known constructively, still a projective point is (potentially) determined. Heyting comments on the need for this provision as follows:

> . . . serious difficulties . . . are caused by the fact that not only points at infinity must be adjoined to the affine plane, but also points for which it is unknown whether they are at infinity or not. [H59, p. 161]

A projective line is determined by two distinct projective points. The definition is based on the lines common to the two projective points; i.e., the lines common to two pencils of lines. For example, in the simplest case, if the two projective points are finite, then these are the pencils of lines through distinct points in the original affine plane, and there is a single common line, connecting these finite points, of which the projective line is an extension. In the case of two distinct pencils of parallel lines; the pencils have no common line, each determines a point at infinity, and the resulting projective line is the line at infinity. Again, even when the status of the original projective points is not known constructively, still a projective line is determined. The distinctive, and perhaps limiting, features of the Heyting extension are the requirements that the construction of a projective point depends on a given pair of distinct finite lines, and the construction of a projective line depends on a pair of distinct projective points previously constructed.

Nearly all the axioms for a projective plane are then verified, although the most essential axiom, which states that two distinct lines have a common point, escapes proof. The axiom considered in [H59] is the weaker version, designated above as the common point property, CPP, involving distinct lines. In [M13], Heyting's axioms for affine geometry are verified for $\mathbb{R}^2$, and a Brouwerian counterexample is given for the Heyting extension, showing that the stronger form of the axiom, SCPP, involving arbitrary lines, is constructively invalid, with the following attempted justification:

> This counterexample concerns the full common point axiom, rather than the limited Axiom P3 as stated in [H59], where only distinct lines are considered. An investigation into the full axiom is necessary for a constructive study based upon numerical meaning, as proposed by Bishop. Questions of distinctness are at the core of constructive problems; any attempted projective extension of the real plane is certain to contain innumerable pairs of lines which may or may not be distinct. [M13, p. 113]

However, taking note of the analytic model $\mathbb{P}^2(\mathbb{R})$, for which CPP is verified, but SCPP is constructively invalid, CPP now appears as a reasonable goal for an extension; thus we have:

Problem. Complete the study of the projective extension of [H59]; verify Heyting's Axiom P3 (CPP), or construct a Brouwerian counterexample.

10 Virtual line extension

Any attempt to build a constructive projective extension of an affine plane encounters difficulties due to the indeterminate nature of arbitrary pencils of lines. Classically, a pencil of lines is either the family of lines passing through a given point, or a family of parallel lines. An example of a family of lines is easily formed from two lines which might be distinct, intersecting or parallel, or might be identical. To obtain the strong common point property, SCPP, in a constructive projective extension, the corresponding pencil must include both these lines, so that it will determine a point of the extension common to both extended lines, whether distinct or not. Thus the definition of pencil must not depend upon a pair of lines previously known to be distinct.

In the projective extension of [M14], the definition of pencil is further generalized; rather than depending upon specific finite lines, it involves the intrinsic properties of a family of lines. Included are pencils of unknown type, with non-specific properties, and pencils for which no lines are known to have been previously constructed.

The definition of line in the extension is independent of the definition of point; it will depend directly upon a class of generalized lines in the finite plane, called virtual lines.

10.1 Definition

The virtual line extension of [M14] is based on a incidence plane $\mathscr{G} = (\mathscr{P}, \mathscr{L})$, consisting of a family $\mathscr{P}$ of points and a family $\mathscr{L}$ of lines, with constructive axioms, definitions, conventions, and results as delineated in [M07].

Definition. Let $\mathscr{G} = (\mathscr{P}, \mathscr{L})$ be an incidence plane.
• For any point $Q \in \mathscr{P}$, define

$$Q^* = \{l \in \mathscr{L} : Q \in l\}.$$

• For any line $l \in \mathscr{L}$, define

$$l^* = \{m \in \mathscr{L} : m \parallel l\}.$$

• A family of lines of the form Q^*, or of the form l^*, is called a *regular pencil.*
• A family of lines α is called a *pencil* if it contains no fewer than two lines, and satisfies the following condition: *If l and m are distinct lines in α with $l, m \in \rho$, where ρ is a regular pencil, then $\alpha \subset \rho$.*
• A pencil of the form Q^* is called a *point pencil.*
• A pencil α with the property that $l \parallel m$, for any lines l and m in α, is called a *parallel pencil.*
[M14, Defn. 2.1]

In the extension, a point pencil Q^*, consisting of all lines through Q, will represent the original finite point Q. A pencil l^*, consisting of all lines parallel to the line l, will result in an infinite point. However, the extension also admits parallel pencils which need not arise from given lines, but which nevertheless result in points at infinity.

10.2 Virtual lines

A problem that arises in the construction of a projective extension is the difficulty in determining the nature of an arbitrary line in the extension, by means of an object in the original plane. If a line λ on the extended plane contains a finite point, then the set λ_f, of all finite points on λ, is a line in the original plane. However, if λ is the line at infinity, then λ_f is void. Since constructively it is in general not known which is the case, we adopt the following:

Definition. A set p of points in $\mathscr{P}$ is said to be a *virtual line* if it satisfies the following condition: *If p is inhabited, then p is a line.* [M14, Defn. 3.1]

Given any virtual lines p and q, one can construct a pencil $\varphi(p, q)$ that contains each of the virtual lines p and q, if it is a line [M14, Thm. 3.4].

The notion of virtual line also helps in resolving a problem that arises in connection with pencils of lines. The family of lines common to two distinct pencils may consist of a single line (as in the case of two point pencils, or a point pencil and a regular parallel pencil), or it may be void (as in the case of two regular parallel pencils); constructively, it is in general unknown which alternative holds. The following definition provides a tool for dealing with this situation.

Definition. For any distinct pencils α and β, define

$$\alpha \sqcap \beta = \{Q \in \mathscr{P} : Q \in l \in \alpha \cap \beta \text{ for some line } l \in \mathscr{L}\}.$$

The set of points $\alpha \sqcap \beta$ is called the *core* of the pair of pencils α, β. [M14, Defn. 3.2]

The core, as a set of finite points (which might be void), is a constructive substitute for a possible line that is common to two pencils.

Theorem. *For any distinct pencils α and β, the core $\alpha \sqcap \beta$ is a virtual line.* [M14, Lm. 3.3]

10.3 Extension

Points of the extension, called *e-points*, are defined using a selected class of pencils of lines, called *prime pencils;* the prime pencil α determines the e-point $\overline{\alpha}$. Lines in the extension are not formed from previously constructed e-points; they are direct extensions of objects in the original plane. Lines of the extension, called *e-lines*, are defined using a selected class of virtual lines, called *prime virtual lines;* the prime virtual line p in the finite plane determines the e-line λ_p in the extended plane.

The projective plane $\mathscr{G}^* = (\mathscr{P}^*, \mathscr{L}^*)$, where $\mathscr{P}^*$ is the family of e-points, and $\mathscr{L}^*$ is the family of e-lines, is the *projective extension* of the incidence plane $\mathscr{G} = (\mathscr{P}, \mathscr{L})$. The axioms of projective geometry are verified for the extension. The following theorems are the main results; the proof outlines will exhibit the symmetry of the construction, and the utility of adopting independent definitions for e-points and e-lines.

Theorem. *On the projective extension $\mathscr{G}^*$ of the plane $\mathscr{G}$, there exists a unique e-line passing through any two distinct e-points.* [M14, Thm. 5.3]

Proof outline. The given e-points $\overline{\alpha}$ and $\overline{\beta}$ originate from pencils α and β; the core $p = \alpha \sqcap \beta$ of these pencils is a virtual line on the finite plane. This virtual line p determines an e-line λ_p in the extension, which passes through both e-points $\overline{\alpha}$ and $\overline{\beta}$.

Theorem. *On the projective extension $\mathscr{G}^*$ of the plane $\mathscr{G}$, any two e-lines have an e-point in common. If the e-lines are distinct, then the common e-point is unique.* [M14, Thm. 5.5]

Proof outline. The given e-lines λ_p and λ_q originate from virtual lines p and q; these virtual lines determine a pencil $\gamma = \varphi(p, q)$ of lines on the finite plane. This pencil γ determines an e-point $\overline{\gamma}$ in the extension, which lies on both e-lines λ_p and λ_q.

Several definitions in [M14] involve negativistic concepts; for example, Definition 2.1 for pencil, and Definition 3.1 for distinct virtual lines. Can this be avoided? Generally in constructive mathematics one tries to avoid negativistic concepts, but perhaps some are unavoidable in constructive geometry; thus we have:

Problem. Modify the virtual line extension so as to avoid negativistic concepts as far as possible.

10.4 The cotransitivity problem

There is what might be called an irregularity of the extension plane $\mathscr{G}^*$, the constructive invalidity of cotransitivity; this is revealed by a Brouwerian counterexample:

Example. On the virtual line projective extension of the plane $\mathbb{R}^2$, the cotransitivity property for e-points is constructively invalid. [M14, p. 705]

Proof. Given any real number c, construct the virtual line

$$p = \{(t, 0) : t \in \mathbb{R} \text{ and } c = 0\} \cup \{(0, t) : t \in \mathbb{R} \text{ and } c \neq 0\}$$

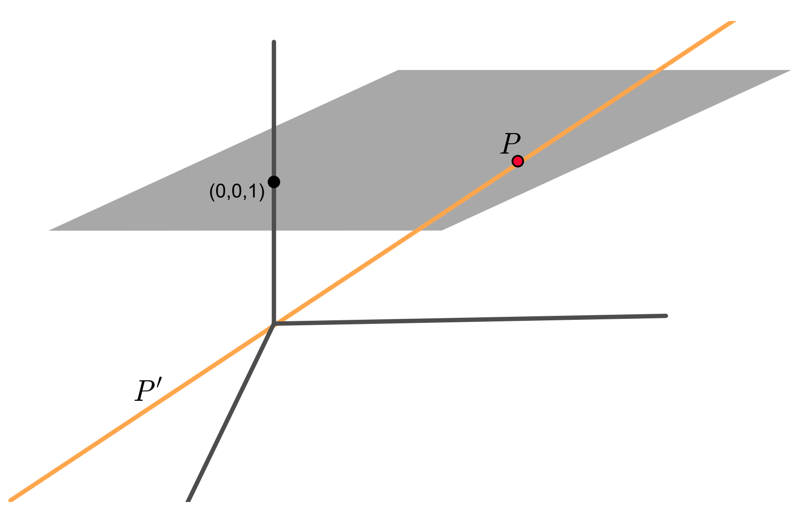
and consider the e-point $\overline{\gamma}$ determined by the pencil $\gamma = \varphi(p, p)$.

Let the x-axis be denoted by l_0; the pencil l_0^* of horizontal lines then determines the e-point $\overline{l_0^*}$. Similarly, we have the y-axis m_0, the pencil m_0^* of vertical lines, and the e-point $\overline{m_0^*}$. By hypothesis, $\overline{\gamma} \neq \overline{l_0^*}$ or $\overline{\gamma} \neq \overline{m_0^*}$. In the first case, suppose that $c = 0$. Then p is the x-axis and $\overline{\gamma} = \overline{l_0^*}$, a contradiction; thus we have $\neg(c = 0)$. In the second case, we find that $c = 0$. Hence WLPO results. □

Problem. Modify the virtual line extension, so that the common point property and cotransitivity are both valid. It is then likely that the strong common point property will not be valid; in that case, provide a Brouwerian counterexample.

11 Analytic extension

The analytic projective plane $\mathbb{P}^2(\mathbb{R})$ described in Section 8 is constructed from subspaces of the linear space $\mathbb{R}^3$, using only constructive properties of the real numbers. This projective plane can be viewed as an extension of the affine plane $\mathbb{R}^2$.



The plane $z = 1$ in $\mathbb{R}^3$ is viewed as a copy of $\mathbb{R}^2$. A point P on the plane $z = 1$ corresponds to the point P' of the extension $\mathbb{P}^2(\mathbb{R})$ that, as a line through the origin in $\mathbb{R}^3$, contains P. A point of $\mathbb{P}^2(\mathbb{R})$, that is a horizontal line through the origin in $\mathbb{R}^3$, is an infinite point of the extension. A line l on the plane $z = 1$ corresponds to the line l' of $\mathbb{P}^2(\mathbb{R})$ that, as a plane through the origin in $\mathbb{R}^3$, contains l. The line of intersection of this plane with the xy-plane is the point at infinity on l'. In this way, $\mathbb{P}^2(\mathbb{R})$ is seen as a projective extension of $\mathbb{R}^2$, with the xy-plane as the line at infinity.

The plane $\mathbb{P}^2(\mathbb{R})$ satisfies both the common point property and cotransitivity. However, as a projective extension of the specific plane $\mathbb{R}^2$, it does not provide an extension of an arbitrary affine plane; thus we have:

Problem. Construct a synthetic projective extension of an arbitrary affine plane, having both the common point property and the cotransitivity property.

References

[B65] E. Bishop, Book Review: *The Foundations of Intuitionistic Mathematics,* by S. C. Kleene and R. E. Vesley, *Bull. Amer. Math. Soc.* 71:850-852, 1965.

[B67] E. Bishop, *Foundations of Constructive Analysis,* McGraw-Hill, New York, 1967.

[B73] E. Bishop, *Schizophrenia in Contemporary Mathematics*, AMS Colloquium Lectures, Missoula, Montana, Amer. Math. Soc., Providence, RI, 1973; reprint in *Contemporary Mathematics* 39:l-32, 1985.

[B75] E. Bishop, The crisis in contemporary mathematics, *Proceedings of the American Academy Workshop on the Evolution of Modern Mathematics*, Boston, 1974*; Historia Math.* 2:507–517, 1975.

[BB85] E. Bishop and D. Bridges, *Constructive Analysis,* Springer-Verlag, Berlin, 1985.

[Be10] M. Beeson, Constructive geometry, *10th Asian Logic Conference,* World Sci. Publ., Hackensack, NJ, 2010, pp. 19–84.

[Be16] M. Beeson, Constructive geometry and the parallel postulate, *Bull. Symb. Log.* 22:1–104, 2016.

[BR87] D. Bridges and F. Richman, *Varieties of Constructive Mathematics,* Cambridge University Press, Cambridge, UK, 1987.

[Brou08] L. E. J. Brouwer, De onbetrouwbaarheid der logische principes, *Tijdschrift voor Wijsbegeerte* 2:152-158, 1908; translation, The unreliability of the logical principles, A. Heyting (ed.), *L. E. J. Brouwer: Collected Works 1: Philosophy and Foundations of Mathematics,* Elsevier, Amsterdam-New York, 1975, pp. 107–111.

[Brou24] L. E. J. Brouwer, Intuitionistische Zerlegung mathematischer Grundbegriffe; reprint, *L. Brouwer, Collected Works, Vol. 1,* North-Holland, Amsterdam, 1975, pp. 275–280.

[BV06] D. Bridges and L. Vîţă, *Techniques of Constructive Analysis,* Springer, New York, 2006.

[Cox55] H. S. M. Coxeter, *The Real Projective Plane, 2nd ed.,* Cambridge University Press, Cambridge, UK, 1955; reprint, Lowe and Brydone, London, 1960.

[Cre73] L. Cremona, *Elementi di geometria projettiva,* G. B. Paravia e Comp., Torino, 1873; translation, C. Leudesdorf, *Elements of Projective Geometry,* Clarendon Press, Oxford, 1985; reprint, Forgotten Books, Hong Kong, 2012.

[Des64] G. Desargues, *Oeuvres De Desargues,* Leiber, 1864.

[H28] A. Heyting, Zur intuitionistischen Axiomatik der projektiven Geometrie, *Math. Ann.* 98:491-538, 1928.

[H59] A. Heyting, Axioms for intuitionistic plane affine geometry, L. Henkin, P. Suppes, A. Tarski (eds.), *The Axiomatic Method, with special reference to geometry and physics: Proceedings of an international symposium held at the University of California, Berkeley, December 26, 1957 - January 4, 1958,* North-Holland, Amsterdam, 1959, pp. 160-173.

[H66] A. Heyting, *Intuitionism: An Introduction,* North-Holland, Amsterdam, 1966.

[Kle72] F. Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen,* A. Deichert, Erlangen, 1872; translation, M. W. Haskell, A comparative review of researches in geometry, *Bull. New York Math. Soc.* 2:215-249, 1893.

[Leh17] D. N. Lehmer, *An Elementary Course in Synthetic Projective Geometry,* Ginn, Boston, 1917.

[LV98] M. Lombard and R. Vesley, A common axiom set for classical and intuitionistic plane geometry, *Ann. Pure Appl. Logic* 95:229-255, 1998.

[M07] M. Mandelkern, Constructive coördinatization of Desarguesian planes, *Beiträge Algebra Geom.* 48:547-589, 2007.

[M13] M. Mandelkern, The common point problem in constructive projective geometry, *Indag. Math. (N.S.)* 24:111-114, 2013.

[M14] M. Mandelkern, Constructive projective extension of an incidence plane, *Trans. Amer. Math. Soc.* 366:691-706, 2014.

[M16] M. Mandelkern, A constructive real projective plane, *J. Geom.* 107:19-60, 2016.

[M18] M. Mandelkern, Constructive harmonic conjugates, *Beiträge Algebra Geom.* 60:391–398, 2019.

[Pam98] V. Pambuccian, Zur konstruktiven Geometrie Euklidischer Ebenen, *Abh. Math. Sem. Univ. Hamburg* 68:7–16, 1998.

[Pam01] V. Pambuccian, Constructive axiomatization of plane hyperbolic geometry, *MLQ Math. Log. Q.* 47:475–488, 2001.

[Pam03] V. Pambuccian, Constructive axiomatization of non-elliptic metric planes, *Bull. Polish Acad. Sci. Math.* 51:49–57, 2003.

[Pam05] V. Pambuccian, Elementary axiomatizations of projective space and of its associated Grassmann space, *Note Mat.* 24:129–141, 2005.

[Pam11] V. Pambuccian, The axiomatics of ordered geometry; I. Ordered incidence spaces, *Expo. Math.* 29:24–66, 2011.

[Pas39] B. Pascal, *Essai pour les coniques,* 1639.

[Pic75] G. Pickert, *Projektive Ebenen, 2. Aufl.,* Springer-Verlag, Berlin-New York, 1975.

[Pon22] J-V. Poncelet, *Traité des Propriétés Projectives des Figures,* Gauthier-Villars, Paris, 1822.

[Ste32] J. Steiner, *Sytematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander,* G. Fincke, Berlin, 1832.

[vDal63] D. van Dalen, Extension problems in intuitionistic plane projective geometry I, II, *Indag. Math.* 25:349-383, 1963.

[vDal96] D. van Dalen, 'Outside' as a primitive notion in constructive projective geometry, *Geom. Dedicata* 60:107-111, 1996.

[vPla95] J. von Plato, The axioms of constructive geometry, *Ann. Pure Appl. Logic* 76:169-200, 1995.

[vPla98] J. von Plato, A constructive theory of ordered affine geometry, *Indag. Math. (N.S.)* 9:549-562, 1998.

[vPla10] J. von Plato, Combinatorial analysis of proofs in projective and affine geometry, *Ann. Pure Appl. Logic* 162:144-161, 2010.

[vSta47] K. G. C. von Staudt, *Geometrie der Lage,* Nürnberg, 1847.

[VY10] O. Veblen and J. W. Young, *Projective Geometry, Vol. 1,* Ginn, Boston, 1910.

[Wei07] C. Weibel, Survey of non-Desarguesian planes, *Notices Amer. Math. Soc.* 54:1294–1303, 2007.

[Y30] J. W. Young, *Projective Geometry,* Math. Assoc. Amer., Open Court, Chicago, 1930.

2019 December 27